Appendix

Appendix A

Nonparametric estimation of $H_g(t)$

Define $N_{ig}(t) = I(X_i \geq t, \delta_i = 1, G = g)$, $i = 1, 2, ...n$, $g \in G$, where G contains all the non-empty subsets of {1,2...k} and $Y_i(t) = I(X_i \leq t)$. For left censored data, we have

$$P(X_i \in (t - dt, t), \delta_i = 1, G = g|F_{t+}) = h_g(t)dt \quad \text{if } X_i \leq t$$
$$= 0 \quad \text{if } X_i > t \tag{A.1}$$

which leads to the fact that,

$$E[dN_{ig}(t)|F_{t+}] = Y_i(t)h_g(t)dt \tag{A.2}$$

where $dN_{ig}(t) = I(X_i = t, \delta_i = 1, J_i = j)$.

Denote $N_g(t) = \sum_{i=1}^{n} N_{ig}(t)$, and $Y(t) = \sum_{i=1}^{n} I(X_i \leq t)$. Now we consider the counting process martingale,

$$M_g(t) = N_g(t) - A_g(t) \tag{A.3}$$

where $A_g(t) = \sum_{i=1}^{n} \int_t^{t0} I(X_i \leq u) h_g(u) du \, and \, t_0 = inf(t; F(t) < 1)$. We also have

$$E[N_g(t)|F_{t+}] = E[A_g(t)|F_{t+}] = A_g(t) \tag{A.4}$$

and

$$E[dA_g(s)|F_{s+}] = E[\text{-}I(X \leq s)|F_{s+}] = dA_g(s). \tag{A.5}$$

From (A.3), (A.4), and (A.5), we have

$$E[dM_g(t)|F_{t+}] = E[dN_g(t) - dA_g(t)|F_{t+}] = 0. \tag{A.6}$$

If $E[dM_g(t)|F_{t+}] = 0$, then for all $t \leq s$

$$\begin{aligned} E[M_g(t)|F_s] - M_g(s) &= E[M_g(t) - M_g(s)|F_s] \\ &= E\left[\int_t^s dM_g(u) \mid F_s\right] \\ &= \int_t^s E\left[E\, dM_g(u) \mid F_{u+} \mid F_s\right] = 0. \end{aligned} \tag{A.7}$$

Thus (A.7) proves that $M_g(t)$ is a martingale. From (A.3) we can write

$$dN_g(t) = Y(t)h_g(t)dt + dM_g(t). \tag{A.8}$$

If $Y(t) > 0$, then we have,

$$\frac{dN_g(t)}{Y(t)} = h_g(t)dt + \frac{dM_g(t)}{Y(t)}. \tag{A.9}$$

If $dM_g(t)$ is noise, then so is $\frac{dM_g(t)}{Y(t)}$, because value of $Y(t)$ at time t are known at time t+. We have $E\left[\frac{dM_g(t)}{Y(t)} \mid F_{t+}\right] = 0.$

Let $C(t) = I(Y(t) > 0)$. Integrating both sides of (A.9) we get

$$\int_t^{t_0} \frac{C(u)dN_g(u)}{Y(u)} = \int_t^{t_0} C(u)h_g(u)du + \int_t^{t_0} \frac{C(u)dM_g(u)}{Y(u)} \tag{A.10}$$

Now from (A.9) the estimator of cumulative reversed hazard rate at time t with g observed as set of possible causes $H_g(t)$ is obtained as

$$\hat{H}_g(t) = \int_t^{to} \frac{C(u)dN_g(u)}{Y(u)} \tag{A.11}$$