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An Application: Representations of Some Systems on Non-Deterministic EEG Signals

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Abstract

In this paper, we introduce the notion of a complex interval which is significant for interval-valued data and interval-based signal processing. First, we present the space of complex intervals and investigate the quasilinear structure of the space of complex intervals. We observe that this space is Hilbert quasilinear space with a set-valued inner product. Finally, we give a application about interval system producing a filter related to interval-valued electroencephalogram (EEG) signals.
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Introduction

Some methods associated with the biostatistic and biometric applications lie in the estimation of accuracy measures for population parameters, change of cancer cells, genetic mutations or EEG signal processing. A few of such applications are bootstrap method, carcinogenesis models and candidate gene analysis. The intervals constitute a mathematical background to eliminate such accuracy datas.

In recent years, there has been increasing interest in interval mathematics and interval-valued functions and their applications. Replacing a precise value by an interval value generally reflects the variability or uncertainty circumstances in observation process. In signal processing, in general, it is very difficult to deal with a process with reliable information about the properties of the expected variations. Such uncertainties in process lead us to set up mathematical foundation of interval-valued data and interval-based signal processing [1-3]. Because of these reasons we want to study the quasilinear structure of the space of complex intervals for research of interval-valued functions (signals).

We mean a function *f* from *R* into the special set I_C by an interval-valued signals. Now let us introduce special properties of I_C. Each element *x* of I_C is called a complex interval, that is,

$$x = \left[\underline{x_r}, \overline{x_r}\right] + i\left[\underline{x_s}, \overline{x_s}\right]$$

where $\left[\underline{x_r}, \overline{x_r}\right]$ and $\left[\underline{x_s}, \overline{x_s}\right]$ are nonempty closed (real) intervals of *R* and *i*=√(-1), the complex unit. The intervals $\left[\underline{x_r}, \overline{x_r}\right]$ and $\left[\underline{x_s}, \overline{x_s}\right]$ are called real and imaginary part of *x*, respectively. For example, *x*=[-2,1]+i[3,7] is a complex interval where $\left[\underline{x_r}, \overline{x_r}\right]=[-2,1]$ and $\left[\underline{x_s}, \overline{x_s}\right]=[3,7]$. Of course, each real interval is a complex interval. $\underline{x_{r(s)}}$ may equal to $\overline{x_{r(s)}}$ and in this case $\left[\underline{x_{r(s)}}, \overline{x_{r(s)}}\right]$ is written as $[x_{r(s)}, x_{r(s)}]$ or $\{x_{r(s)}\}$ or only $x_{r(s)}$ and is called a degenerate complex interval. More clearly, for any $x,y \in R$, $[x,x]+i[y,y]$ is a complex degenerate interval and is written as $\{x\} + i\{y\}$. In this respect, *R* is a subset of I_R, the set of all nonempty closed (real) intervals and the complex number C is a subset of I_C. Further, I_R is a subset of I_C.

To get a comprehensive and healthy interval-valued signal processing we need a mathematical point of view. This is due to the fact that the sets I_R and I_C are not vector spaces. However, they are in accord with a similar (in fact more general) structure, namely a quasilinear space.

Inner Product Quasilinear Spaces

We will start this section by giving the definition of quasilinear spaces and some basic notions which will be used later on. Note that quasilinear spaces has been only introduced on the reel field R so far. As distinct from Aseev's definition [4] and from our some previous studies, we will consider the quasilinear spaces over a general field K. The elements of K are called scalars, practically, they will be real or complex numbers. We think that this approach may be formed more useful and suitable backdrop for some applications, especially, for interval-valued data analysis and signal processing.

A set X is called a quasilinear space, (briefly QLS) on field K, if a partial order relation "≤", an algebraic sum operation, and an operation of multiplication by real or complex numbers are defined in it in such a way that the following conditions hold for any elements $x,y,z,v \in X$ and any $\alpha,\beta \in$ K [4]:

$$x \leq x, \quad (1)$$

$$x \leq z \text{ if } x \leq y \text{ and } y \leq z, \quad (2)$$

$$x = y \text{ if } x \leq y \text{ and } y \leq x, \quad (3)$$

$$x + y = y + x, \quad (4)$$

$$x + (y + z) = (x + y) + z, \quad (5)$$

there exists anelement $\theta \in X$ such that $x + \theta = x$,(6)

$$\alpha(\beta x) = (\alpha\beta)x, \quad (7)$$

$$\alpha(x + y) = \alpha x + \alpha y, \quad (8)$$

$$1x = x, \quad (9)$$

$$0x = \theta, \quad (10)$$

$$(\alpha + \beta)x \leq \alpha x + \beta x, \quad (11)$$

$$x + z \leq y + v \text{ if } x \leq y \text{ and } z \leq v, \quad (12)$$

$$\alpha x \leq \alpha y \text{ if } x \leq y. \quad (13)$$

K is called the scalar field of the quasilinear space X, and X is called a real quasilinear space if K = R and is called a complex quasilinear space if K = C. Mostly K will be C in this work.

Any real linear space is a QLS with the partial order relation defined by "$x \leq y$ if and only if $x = y$". In this case, QLS axioms is the linear space axioms.

Perhaps the most popular example of nonlinear real QLSs is I_R with the inclusion relation "⊆", with algebraic sum operation

$$x + y = [\underline{x}, \overline{x}] + [\underline{y}, \overline{y}] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}] = \{a + b : a \in x, b \in y\}$$

and with real-scalar multiplication

$$\lambda x = \lambda[\underline{x}, \overline{x}] = \begin{cases} [\lambda\underline{x}, \lambda\overline{x}], & \lambda \geq 0 \\ [\lambda\overline{x}, \lambda\underline{x}], & \lambda < 0 \end{cases} = \{\lambda a : a \in x\}.$$

Proof of this assertion will be given by more general form. Another name of I_R is $\Omega_C(R)$, the set of all nonempty compact convex subsets of real numbers. A compact convex subset of R^n is called a convex body and the space of this subsets is denoted by $\Omega_C(R^n)$, $n = 1,2,\ldots$. Further, $\Omega(R^n)$ is the set of all nonempty compact subsets of real numbers and is another important example of nonlinear real QLSs. In general, $\Omega(E)$ and $\Omega_C(E)$ stand for families of all nonempty closed bounded and nonempty convex closed bounded subsets of any normed linear space E, respectively. Both are real or complex QLSs with the inclusion relation, with multiplication by a real or complex number λ which is defined by

$$\lambda A = \{\lambda a : a \in A\}$$

and with a slight modification of addition as follows:

$$A + B = \overline{\{a + b : a \in A, b \in B\}}$$

where the closure is taken on the norm topology of E.

We should mention $\mathbb{I}_{\mathbb{R}}^{n}$, the space of all n-fold real interval vectors [5] where n is a positive integers. For example a typical element of $\mathbb{I}_{\mathbb{R}}^{2}$ can be given as ([-1,2],[3,6]). Further, $\mathbb{I}_{\mathbb{C}}^{n}$ is the space of all n-fold complex interval vectors ([1,3]-2i[1,2],[-1,0]+i[1,5]) is a typical element of $\mathbb{I}_{\mathbb{C}}^{2}$.

Remark 2.1 For n=2,3,..., $\mathbb{I}_{\mathbb{R}}^{n}$ is different from $\Omega_C(R^n)$. For example, the unit ball

$$B=\left\{x=\left(x_1,x_2\right):x_1^2+x_2^2\leq 1\right\}$$

of R^2 is an element of $\Omega_C(R^2)$. But B is not an element of $\mathbb{I}_{\mathbb{R}}^{2}$. Further, they both are nonlinear quasilinear spaces by different operations and relations.

In this work we mainly interested in algebraic and metric structure of n-fold real or, in general, complex interval vectors and the interval functions. Although, they are not a vector space we will choice the term interval-vector due to its conventional usage.

Now, let us record some basic necessary results from [4]. In a QLS X, the element θ is minimal, i.e., $x = \theta$ if $x \leq \theta$. An element x' is called inverse of $x \in X$ if $x + x' = \theta$. The inverse is unique whenever it exists. An element x possessing inverse is called *regular*, otherwise is called *singular*.

Lemma 2.1 *[4] Suppose that each element x in QLS X has inverse element x′ ∈ X. Then the partial order in X is determined by equality, the distributivity conditions hold, and consequently X is a linear space.*

Hence in a real linear space, the equality is the only way to define a partial order such that conditions (1)-(13) hold.

It will be assumed in what follows that $-x = (-1)\cdot x$. Also, note that $-x$ may not be x'. Any element x in a QLS is regular if and only if $x - x = \theta$ if and only if $x' = -x$.

Definition 2.1 *Suppose that X is a QLS and Y ⊆ X. Then Y is called a subspace of X whenever Y is a QLS with the same partial order on X.*

Theorem 2.1 *Y is subspace of QLS X if and only if for every x, y ∈ Y and α, β ∈ K, α.x + β.y ∈ Y.*

Proof of this theorem is quite similar to its classical linear algebraic analogue.
So we see easily that I_R is a subspace of I_C. Further, I_C is a subspace of $\Omega(C)$.

Let X be a QLS and Y be a subspace of X. Suppose that each element x in Y has inverse element $x' \in Y$ then by Lemma 2.1 the partial order on Y is determined by the equality. In this case Y is a linear subspace of X. An element x in QLS X is said to be *symmetric* if $-x = x$ and X_{sym} denotes the set of all *symmetric* elements. Also, X_r stands for the set of all regular elements of X while X_s stands for the sets of all singular elements and zero in X. Further, it can be easily shown that X_r, X_{sym} and X_s are subspaces of X. They are called *regular*, *symmetric* and *singular subspaces* of X, respectively. Furthermore, it isn't hard to prove that summation of a regular element with a singular element is a singular element and the regular subspace of X is a linear space while the singular one is nonlinear at all.

Let X be a real or complex QLS. The real-valued function on X is called a *norm*, if the following conditions hold [4-9]:

$$\|x\| > 0 \text{ if } x \neq 0, \tag{14}$$

$$\|x+y\| \leq \|x\| + \|y\|, \tag{15}$$

$$\|\alpha x\| = |\alpha| \|x\|, \tag{16}$$

$$\text{if } x \leq y, \text{then } \|x\| \leq \|y\|, \tag{17}$$

$$\text{if for any } \varepsilon>0 \text{ there exists an element } x_\varepsilon \in X \text{ such that} \tag{18}$$

$$x \leq y + x_\varepsilon \text{ and } \|x\| \leq \varepsilon \text{ then } x \leq y,$$

here x,y,x_ε are arbitrary element in X and α is any scalar.

A quasilinear space X with a norm defined on it, is called *normed quasilinear space* (briefly, *normed QLS*). It follows from Lemma 2 that if any x ∈ X has inverse element $x' \in X$, then the concept of normed QLS coincides with the concept of real normed linear space. Notice again that x' may not be exist but if x' exists then $x' = -x$. Hausdorff metric or norm metric on X is defined by the equality

$$h(x,y) = \inf\{r \geq 0 : x \leq y + a_1^{(r)}, y \leq x + a_2^{(r)} \text{and} \|a_i^{(r)}\| \leq r, i = 1,2\}.$$

Since $x \leq y+(x-y)$ and $y \leq x+(y-x)$, the quantity $h(x,y)$ is well-defined for any elements $x,y \in X$, and it is not hard to see that the function h satisfies all the metric axioms. Also we should note that $h(x,y)$ may not equal to $\|x-y\|$ if X is not a linear space; however $h(x,y) \leq \|x-y\|$ for every $x, y \in X$.

Lemma 2.2 *[4] The operations of algebraic sum and multiplication by real or complex numbers are continuous with respect to the Hausdorff metric. The norm is continuous function respect to the Hausdorff metric.*

Proposition 2.1 The following conditions hold with respect to the Hausdorff metric:

$$i)\ h(\alpha x,\alpha y)=|\alpha|h(x,y)$$
$$ii)\ h(x+y,z+v)\le h(x,z)+h(y,v),$$
$$iii)\ \|x\|=h(x,\theta)$$

for each $\alpha \in K$ and every $x,y,z,v \in X$.

Lemma 2.3 *[4]*

(a) Suppose that $x_n \to x_0$ and $y_n \to y_0$, and that $x_n \le y_n$ for any positive integer n. Then $x_0 \le y_0$.
(b) Suppose that $x_n \to x_0$ and $z_n \to x_0$. If $x_n \le y_n \le z_n$ for any positive integer n, then $y_n \to x_0$.
(c) If $x_n + y_n \to x_0$ and $y_n \to \theta$ then $x_n \to x_0$.

Example 2.1 *[4] For a normed linear space E, a norm on Ω(E) is defined by $\|A\|_{\Omega(E)}=\sup_{a\in E}\|a_E\|$. Hence $\Omega_C(E)$ and Ω(E) are normed QLSs. In this case the Hausdorff (norm) metric is defined as usual:*

$$h(x,y)=\inf\{r\ge 0: x\subseteq y+S_r(\theta),\ y\subseteq x+S_r(\theta)\},$$

where $S_r(\theta)$ is closed ball of E and x,y are elements of $\Omega_C(E)$ or Ω(E). Further, $\Omega_C(E)$ is a closed subspace of Ω(E). Further, Ω(E) and $\Omega_C(E)$ are Banach space. For $E=R$, $\Omega_C(R)=I_R$ is a Banach space with the norm defined by $[\underline{x},\overline{x}]_{\mathbb{I}_R}=\max_{a\in[\underline{x},\overline{x}]}|a|$.

Now, let us give a useful type of QLSs called consolidate QLS.

Definition 2.2 *[10] Let X be a QLS, M ⊆ X and x ∈ M. The set*

$$F_x^M=\{z\in M_r : z\le x\}$$

is called *floor in M* of x . In the case of $M = X$ it is called only *floor of x* and written briefly F_x instead of F_x^X.

Floor of an element x in linear spaces is $\{x\}$. Therefore, it is nothing to discuss the notion of floor of an element in a linear space.

Definition 2.3 *[10] Let X be a qls and M ⊆ X. Then the union set*

$$U_{x\in M} F_x^M$$

is called *floor of M* and is denoted by F_M. In the case of $M = X$, F_X is called floor of the qls X.
On the other hand, the set

$$\mathcal{F}_M^X = U_{x\in M} F_x^M$$

is called *floor in X* of M and is denoted by $\mathcal{F}_M^X$.

We refer to the reader to [10] for detailed informations about this topic.

Definition 2.4 *[11] A quasilinear space X is called consolidate QLS whenever $\sup F_y$ exists for every y ∈ X and*

$$y=\sup F_y=\sup\{z\in X_r : z\le y\}.$$

Otherwise, X is called *non consolidate QLS* and in this case it is written as *nc-QLS* sake for brevity.

Especially, we should note that the supremum in this definition is considered according to relation "≤" on X.

Example 2.2 *[11] For any normed linear space E, Ω(E) and $\Omega_C(E)$ are consolidate NQLS.*
On the other hand, it is clear that $(\Omega_C(R))_s \cup\{0\}$ is nc-QLS. For example,

$$\sup\{x : x\in((\Omega_C(\quad))_s\cup\{0\})_r, x\subseteq y\}=\{0\}\ne y$$

for element $y=[-2,3]\in(\Omega_C(R))_s\cup\{0\}$. Also, no there exists any element x such that $x \subseteq z$ for $z=[1,3] \in (\Omega_C(R))_s\cup\{0\}$.

Let us now give the definition of inner-product in a quasilinear space which is consistent with its linear analogue [7-9]. Later we will present some fundamental properties of inner-product and Hilbert QLSs. Previously let us introduce a definition.

Definition 2.5 *For two quasilinear spaces (X,≤) and (Y,≥), Y is called compatible contains X whenever X ⊆ Y and the partial order relation "≤" on X is the restriction of the partial order relation "≥" on Y. We briefly use the symbol $X \sqsubseteq Y$ in this case. We write $X \equiv Y$ whenever $X \sqsubseteq Y$ and $Y \sqsubseteq X$.*

Remark 2.2 *Hence $X \equiv Y$ means X and Y are the same sets with the same partial order relations which make them quasilinear spaces. However, we may write $X = Y$ for $X \equiv Y$ whenever the relations are clear from context.*

Definition 2.6 *Let X be a QLS. Consolidation of X is the smallest consolidate QLS $\hat{X}$ which compatible contains X, that is, if there exists another consolidate QLS Y which compatible contains X then $\hat{X} \subseteq Y$.*

Clearly, $\hat{X} = X$ for some consolidate QLS X. We do not know yet whether each QLS has a consolidation. This notion is unnecessary in consolidate QLSs, hence is in linear spaces. Further, $\Omega_C\,\widehat{(\mathbb{R}^n)_s} = \Omega_C(R^n)$.

For a QLS X, the set

$$F_y^{\hat{X}} = \left\{ z \in \left(\hat{X}\right)_r : z \le y \right\}.$$

is *the floor of y in* $\hat{X}$.

Now, let us give an extended definition of inner-product given in [8]. We can say that the inner product in the following definition may be seen a set-valued inner product on quasilinear spaces.

Definition 2.7 *Let X be a quasilinear space having a consolidation $\hat{X}$. A mapping $\langle , \rangle : X \times X \to \Omega(K)$ is called an inner product on X if for any $x,y,z \in X$ and $\alpha \in K$ the following conditions are satisfied :*

If $x,y \in X_r$ then $\langle x,y \rangle \in \Omega_C(K)_r \equiv K$, (19)

$\langle x+y,z \rangle \subseteq \langle x,z \rangle + \langle y,z \rangle$, (20)

$\langle \alpha x,y \rangle = \alpha \langle x,y \rangle$, (21)

$\langle x,y \rangle = \langle y,x \rangle$, (22)

$\langle x,x \rangle \ge 0$ for $x \in X_r$ and $\langle x,x \rangle = 0 \Leftrightarrow x = 0$, (23)

$\|\langle x,y \rangle\|_{\Omega(K)} = \sup\{\|\langle a,b \rangle\|_{\Omega(K)} : a \in F_x\hat{X}, b \in F_y\hat{X}\}$, (24)

if $x \le y$ and $u \le v$ then $\langle x,u \rangle \subseteq \langle y,v \rangle$, (25)

if for any $\varepsilon > 0$ there exists an element $x_\varepsilon \in X$ such that (26)
$x \le y + x_\varepsilon$ and $\langle x_\varepsilon, x_\varepsilon \rangle \subseteq S_\varepsilon(\theta)$ then $x \le y$.

A quasilinear space with an inner product is called an inner product quasilinear space, briefly, IPQLS.

Remark 2.3 *For some $x \in X_r$, $\langle x, x \rangle \ge 0$ means $\langle x, x \rangle$ is non-negative, that is, the order "$\ge$" in the definition is the usual order on $\Omega_C(K)_r \equiv K$. It should not be confused with the order "$\le$" on X.*

Example 2.3 *[7, 9] Let X be a linear Hilbert space. Then the space $\Omega(X)$ is a Hilbert quasilinear space by the inner-product defined by*

$$\langle A,B \rangle_{\Omega(X)} = \overline{\left\{ a, b_X : a \in A, b \in B \right\}} \quad \text{for } A,B \in \Omega(X).$$

Every IPQLS X is a normed QLS with the norm defined by

$$x = \sqrt{\|\langle x,x \rangle\|_{\Omega(R)}}$$

for every $x \in X$. This norm is called inner-product norm. Classical norm of I_R [4] is generated by the above inner-product. Further $x_n \to x$ and $y_n \to y$ in a IPQLS then $\langle x_n, y_n \rangle \to \langle x,y \rangle$.

A IPQLS is called *Hilbert QLS,* if it is complete according to the inner-product (norm) metric. For example, I_R is a *Hilbert QLS*.

Quasilinear Structure of the Space of Complex Intervals

Let us impose an order relation on I_C:

$$x \le y \text{ if and only if } \left[\underline{x_r}, \overline{x_r}\right] \subseteq \left[\underline{y_r}, \overline{y_r}\right] \text{and} \left[\underline{x_s}, \overline{x_s}\right] \subseteq \left[\underline{y_s}, \overline{y_s}\right]$$

for $x,y \in I_C$. Then "≤" is a partial order relation on I_C. Further, consider following operations:

$$x+y=\left[\underline{x_r},\overline{x_r}\right]+i\left[\underline{x_s},\overline{x_s}\right]+\left[\underline{y_r},\overline{y_r}\right]+i\left[\underline{y_s},\overline{y_s}\right]$$
$$=\left[\underline{x_r}+\underline{y_r},\overline{x_r}+\overline{y_r}\right]+i\left[\underline{x_s}+\underline{y_s},\overline{x_s}+\overline{y_s}\right]$$
$$=\left\{a+ib: a\in\left[\underline{x_r}+\underline{y_r},\overline{x_r}+\overline{y_r}\right], b\in\left[\underline{x_s}+\underline{y_s},\overline{x_s}+\overline{y_s}\right]\right\} \quad \text{and}$$
$$\lambda\cdot x=\lambda\cdot\left[\underline{x_r},\overline{x_r}\right]+i\left(\lambda\cdot\left[\underline{x_s},\overline{x_s}\right]\right)$$
$$=\left\{\lambda a+i\lambda b: a\in\left[\underline{x_r},\overline{x_r}\right], b\in\left[\underline{x_s},\overline{x_s}\right]\right\}$$

on I_C where $i=\sqrt{(-1)}$ and $\lambda \in C$.

Theorem 3.1 I_C *is a quasilinear space on the field C by the above relation and algebraic operations.*

Proof. Verification of first five axioms to be a QLS is too straighforward. Further, the degenerate interval $\theta = [0,0] = \{0\}$, is the identity element of the addition. Further, for $1,0 \in C$ and $x \in I_C$, obviously, $1\cdot x = x$, $0\cdot x = \theta$, and easily see that $\alpha\cdot(\beta x) = (\alpha\beta)\cdot x$, and $\alpha\cdot(x+y) = \alpha\cdot x + \alpha\cdot y$. For $\alpha,\beta \in C$ and $x \in I_C$, $(\alpha+\beta)x \le \alpha x + \beta x$,

Let us determine regular and singular subspace of I_R and I_C. If $x = [a,a] = \{a\}$, $a \in R$ is a degenerate interval, then x is regular and there is no regular element other then such types. Hence

$$\left(\mathbb{I}_{\mathbb{R}}\right)_r=\left\{\{a\}: a\in\mathbb{R}\right\} \quad \text{and}$$
$$\left(\mathbb{I}_{\mathbb{R}}\right)_s=\{0\}\cup\{[a,b]: a,b\in\mathbb{R} \text{ and } a<b\}.$$

An easy observation shows $\left(\mathbb{I}_{\mathbb{C}}\right)_r=\left\{\{a+ib\}: a,b\in\mathbb{R}\right\}$

and

$$\left(\mathbb{I}_{\mathbb{C}}\right)_s=\{0\}\cup\{\left[\underline{x_r},\overline{x_r}\right]+i\left[\underline{x_s},\overline{x_s}\right]: \underline{x_r}<\overline{x_r} \text{ and } \underline{x_s}<\overline{x_s}\}.$$

Regular subspaces of I_R and I_C just are, or can be identified with, R and C, respectively.

We can impose a lot of norm on I_C. For example,

$$\|x\|_1=\left\|\left[\underline{x_r},\overline{x_r}\right]+i\left[\underline{x_s},\overline{x_s}\right]\right\|_1=\left\|\left[\underline{x_r},\overline{x_r}\right]\right\|_{\mathbb{I}_{\mathbb{R}}}+\left\|\left[\underline{x_s},\overline{x_s}\right]\right\|_{\mathbb{I}_{\mathbb{R}}}$$
$$=\sup_{a\in\left[\underline{x_r},\overline{x_r}\right]}|a|+\sup_{a\in\left[\underline{x_s},\overline{x_s}\right]}|a|$$

is a norm on I_C. Perhaps, the more useful norm on I_C is

$$\|x\|_2=\left\|\left[\underline{x_r},\overline{x_r}\right]+i\left[\underline{x_s},\overline{x_s}\right]\right\|_2=\sqrt{\left\|\left[\underline{x_r},\overline{x_r}\right]\right\|_{\mathbb{I}_{\mathbb{R}}}^2+\left\|\left[\underline{x_s},\overline{x_s}\right]\right\|_{\mathbb{I}_{\mathbb{R}}}^2}.$$

Last we should note the max-norm

$$\|x\|_\infty=\left\|\left[\underline{x_r},\overline{x_r}\right]+i\left[\underline{x_s},\overline{x_s}\right]\right\|_\infty=\max\left(\left\|\left[\underline{x_r},\overline{x_r}\right]\right\|_{\mathbb{I}_R},\left\|\left[\underline{x_s},\overline{x_s}\right]\right\|_{\mathbb{I}_R}\right)$$

is another important norm on I_C.

Let us only verify the last two conditions to be norm for $\|\cdot\|_2$. Let $x=\left[\underline{x_r},\overline{x_r}\right]+i\left[\underline{x_s},\overline{x_s}\right]$ and $y=\left[\underline{y_r},\overline{y_r}\right]+i\left[\underline{y_s},\overline{y_s}\right]$ be arbitrary elements of I_C. If $x°y$, then $\left[\underline{x_r},\overline{x_r}\right]\subseteq\left[\underline{y_r},\overline{y_r}\right]$ and $\left[\underline{x_s},\overline{x_s}\right]\subseteq\left[\underline{y_s},\overline{y_s}\right]$. This implies $\sup_{a\in\left[\underline{x_r},\overline{x_r}\right]}|a|=\left[\underline{x_r},\overline{x_r}\right]_{\mathbb{I}_{\mathbb{R}}}\le\sup_{a\in\left[\underline{y_r},\overline{y_r}\right]}|a|=\left[\underline{y_r},\overline{y_r}\right]_{\mathbb{I}_{\mathbb{R}}}$

and $\sup_{a\in\left[\underline{x_r},\overline{x_r}\right]}|a|=\left\|\left[\underline{x_r},\overline{x_r}\right]\right\|_{\mathbb{I}_{\mathbb{R}}}\le\sup_{a\in\left[\underline{y_r},\overline{y_r}\right]}|a|=\left\|\left[\underline{y_r},\overline{y_r}\right]\right\|_{\mathbb{I}_{\mathbb{R}}}$

Further, $\|x\|_2=\sqrt{\left\|\left[\underline{x_r},\overline{x_r}\right]\right\|_{\mathbb{I}_{\mathbb{R}}}^2+\left\|\left[\underline{x_s},\overline{x_s}\right]\right\|_{\mathbb{I}_{\mathbb{R}}}^2}\le\sqrt{\left\|\left[\underline{y_r},\overline{y_r}\right]\right\|_{\mathbb{I}_{\mathbb{R}}}^2+\left\|\left[\underline{y_s},\overline{y_s}\right]\right\|_{\mathbb{I}_{\mathbb{R}}}^2}=\|y\|_2$.

For the last conditions of the norm, let $\varepsilon>0$ be arbitrary and there exists an element $x_\varepsilon=\left[\underline{x_{\varepsilon r}},\overline{x_{\varepsilon r}}\right]+i\left[\underline{x_{\varepsilon s}},\overline{x_{\varepsilon s}}\right]\in\mathbb{I}_C$ such that $x°y+x_\varepsilon$ and $\|x_\varepsilon\|_2\le\varepsilon$. The conditions indicates $\left[\underline{x_r},\overline{x_r}\right]\subseteq\left[\underline{y_r},\overline{y_r}\right]+\left[\underline{x_{\varepsilon r}},\overline{x_{\varepsilon r}}\right]$ and $\left[\underline{x_s},\overline{x_s}\right]\subseteq\left[\underline{y_s},\overline{y_s}\right]+\left[\underline{x_{\varepsilon s}},\overline{x_{\varepsilon s}}\right]$,

$$\text{and} \quad \left\|\left[\underline{x_{\varepsilon r}}, \overline{x_{\varepsilon r}}\right]\right\|_{\mathbb{I}_R}^2 + \left\|\left[\underline{x_{\varepsilon s}}, \overline{x_{\varepsilon s}}\right]\right\|_{\mathbb{I}_R}^2 \le \varepsilon^2.$$

Now let us assume x≰y with the above conditions. This means $\left[\underline{x_r}, \overline{x_r}\right] \subsetneq \left[\underline{y_r}, \overline{y_r}\right]$ or $\left[\underline{x_s}, \overline{x_s}\right] \subsetneq \left[\underline{y_s}, \overline{y_s}\right]$. Suppose $\left[\underline{x_r}, \overline{x_r}\right]$Ú$\left[\underline{y_r}, \overline{y_r}\right]$. This gives a real number $a \in \left[\underline{x_r}, \overline{x_r}\right]$ such that $a \notin \left[\underline{y_r}, \overline{y_r}\right]$. Since $\left[\underline{y_r}, \overline{y_r}\right]$ is closed

$$h(a, \left[\underline{y_r}, \overline{y_r}\right]) = \inf_{b \in \left[\underline{y_r}, \overline{y_r}\right]} |a-b| \neq 0.$$

By the hypothesis, for $\varepsilon = \dfrac{h(a, \left[\underline{y_r}, \overline{y_r}\right])}{2}$,

there exists an element $\left[\underline{x_{\varepsilon r}}, \overline{x_{\varepsilon r}}\right] \in \mathbb{I}_{\mathbb{R}}$ such that $\left[\underline{x_r}, \overline{x_r}\right] \subseteq \left[\underline{y_r}, \overline{y_r}\right] + \left[\underline{x_{\varepsilon r}}, \overline{x_{\varepsilon r}}\right]$ and $\left\|\left[\underline{x_{\varepsilon r}}, \overline{x_{\varepsilon r}}\right]\right\|_{\mathbb{I}_R} \le \|x\|_2 = \sqrt{\left\|\left[\underline{x_{\varepsilon r}}, \overline{x_{\varepsilon r}}\right]\right\|_{\mathbb{I}_R}^2 + \left\|\left[\underline{x_{\varepsilon s}}, \overline{x_{\varepsilon s}}\right]\right\|_{\mathbb{I}_R}^2} \le \varepsilon.$

Thus $a \in \left[\underline{y_r}, \overline{y_r}\right] + \left[\underline{x_{\varepsilon r}}, \overline{x_{\varepsilon r}}\right]$ since $a \in \left[\underline{x_r}, \overline{x_r}\right]$. Then we can find $b \in \left[\underline{y_r}, \overline{y_r}\right]$ and $a_\varepsilon \in \left[\underline{x_{\varepsilon r}}, \overline{x_{\varepsilon r}}\right]$ such that

$$a = b + a_\varepsilon.$$

Hence

$$\begin{aligned}
0 &= |a-(b+a_\varepsilon)| \ge \big||a-b| - |a_\varepsilon|\big| \\
&\ge \left|h(a, \left[\underline{y_r}, \overline{y_r}\right]) - |a_\varepsilon|\right| \\
&\ge \left|h(a, \left[\underline{y_r}, \overline{y_r}\right]) - \frac{h\left(a, \left[\underline{y_r}, \overline{y_r}\right]\right)}{2}\right| \\
&= \left|\frac{h(a, \left[\underline{y_r}, \overline{y_r}\right])}{2}\right| = \varepsilon
\end{aligned}$$

This is a contradiction. Thus $\left[\underline{x_r}, \overline{x_r}\right] \subseteq \left[\underline{y_r}, \overline{y_r}\right]$. If the case $\left[\underline{x_s}, \overline{x_s}\right]$Ú$\left[\underline{y_s}, \overline{y_s}\right]$ is valid then by the similar way we get $\left[\underline{x_s}, \overline{x_s}\right] \subseteq \left[\underline{y_s}, \overline{y_s}\right]$. Consequently $(\mathbb{I}_{\mathbb{C}}, \|x\|_2)$ is a normed QLS.

On the other hand, it can be easily seen that

$$\sup\{y \in (\mathbb{I}_{\mathbb{C}})_r : y \subseteq \left[\underline{x_r}, \overline{x_r}\right] + i\left[\underline{x_s}, \overline{x_s}\right]\} = \left[\underline{x_r}, \overline{x_r}\right] + i\left[\underline{x_s}, \overline{x_s}\right]$$

for any element $x = \left[\underline{x_r}, \overline{x_r}\right] + i\left[\underline{x_s}, \overline{x_s}\right]$ of I_C. Hence I_C is a consolidate QLS.

Now we shall present very important theorem for the space of complex intervals.

Theorem 3.2 *The space I_C is inner product quasilinear space with Ω(C)-valued inner-product function defined by*

$$\begin{aligned}
\langle x, y\rangle &= \left\langle \left[\underline{x_r}, \overline{x_r}\right] + i\left[\underline{x_s}, \overline{x_s}\right], \left[\underline{y_r}, \overline{y_r}\right] + i\left[\underline{y_s}, \overline{y_s}\right]\right\rangle \\
&= \left[\underline{x_r}, \overline{x_r}\right]\left[\underline{y_r}, \overline{y_r}\right] + \left[\underline{x_s}, \overline{x_s}\right]\left[\underline{y_s}, \overline{y_s}\right] + i(\left[\underline{x_s}, \overline{x_s}\right]\left[\underline{y_r}, \overline{y_r}\right] - \left[\underline{x_r}, \overline{x_r}\right]\left[\underline{y_s}, \overline{y_s}\right])
\end{aligned} \quad (27)$$

for elements $x = \left[\underline{x_r}, \overline{x_r}\right] + i\left[\underline{x_s}, \overline{x_s}\right], y = \left[\underline{y_r}, \overline{y_r}\right] + i\left[\underline{y_s}, \overline{y_s}\right]$ where the multiplication of real intervals is in its usual sense [5].

Proof. Since sum, difference and product operations of intervals is closed [5], we have that $\left[\underline{x_r}, \overline{x_r}\right]\left[\underline{y_r}, \overline{y_r}\right] + \left[\underline{x_s}, \overline{x_s}\right]\left[\underline{y_s}, \overline{y_s}\right] \in \mathbb{I}_{\mathbb{R}}$ and $\left[\underline{x_s}, \overline{x_s}\right]\left[\underline{y_r}, \overline{y_r}\right] - \left[\underline{x_r}, \overline{x_r}\right]\left[\underline{y_s}, \overline{y_s}\right] \in \mathbb{I}_{\mathbb{R}}$. Thus, the equality (27) is well-defined, since $x, y \in \mathbb{I}_{\mathbb{C}}$ and $\mathbb{I}_{\mathbb{C}}$ is a subspace of Ω(C). Let us show that this equality provides inner-product axioms:

- For the regular elements $x = \{a\} + i\{b\}$ and $y = \{c\} + i\{d\}$, $a,b,c,d \in \mathbb{R}$ we write

$$\begin{aligned}
\langle x, y\rangle &= \langle \{a\} + i\{b\}, \{c\} + i\{d\}\rangle \\
&= \{a\}\{c\} + \{b\}\{d\} + i(\{b\}\{c\} - \{a\}\{d\})
\end{aligned}$$

By the first condition of inner product on $\mathbb{I}_{\mathbb{R}}$ we say that $\{a\}\{c\}, \{b\}\{d\}, \{b\}\{c\}, \{a\}\{d\} \in \mathbb{I}_{\mathbb{R}} \equiv \mathbb{R}$ and so $\langle x,y\rangle \in \Omega_C(C)_r \equiv C$ for any $\langle x,y\rangle \in \Omega_C(C)_r \equiv C$.

Hereafter, let us take $[\underline{x_r},\overline{x_r}]=A,[\underline{x_r},\overline{x_r}]=A,[\underline{y_r},\overline{y_r}]=C,[\underline{y_s},\overline{y_s}]=D,[\underline{z_r},\overline{z_r}]=E[\underline{z_s},\overline{z_s}]=F$ for elements

$x=[\underline{x_r},\overline{x_r}]+i[\underline{x_s},\overline{x_s}], y=[\underline{y_r},\overline{y_r}]+i[\underline{y_s},\overline{y_s}]$ *and* $z=[\underline{z_r},\overline{z_r}]+i[\underline{z_s},\overline{z_s}]$ For any $x,y,z \in I_C$

$$\begin{aligned}&\langle x+y,z\rangle=\langle (A+iB)+(C+iD),E+iF\rangle\\&=\langle A+C+i(B+D),E+iF\rangle\\&=(A+C)E+(B+D)F+i[(B+D)E-(A+C)F]\end{aligned}$$

Since $A,B,C,D,E,F\in\Omega_C(R)$ *and* I_R *is a inner-product space,*

$$\begin{aligned}&\langle x+y,z\rangle=(A+C)E+(B+D)F+i[(B+D)E-(A+C)F]\\&\subseteq AE+CE+BF+DF+i(BE+DE-AF-CF)\\&=[(AE+BF)+i(BE-AF)]+[(CE+DF)+i(DE-CF)]\\&=\langle x,z\rangle+\langle y,z\rangle\end{aligned}$$

- For $x,y \in I_C$ and $\alpha\in C$,

$$\begin{aligned}&\langle \alpha x,y\rangle=\langle \alpha(A+iB),C+iD\rangle\\&=(\alpha A)C+(\alpha B)D+i[(\alpha B)C-(\alpha A)D]\\&=\alpha(AC)+\alpha(BD)+i[\alpha(BC)-\alpha(AD)]\\&=\alpha AC+BD+i(BC-AD)]\\&=\alpha\langle x,y\rangle\end{aligned}$$

- *For* $x,y \in I_C$,

$$\begin{aligned}&\langle x,y\rangle=\langle A+iB,C+iD\rangle\\&=AC+BD+i(CB-AD)\\&=CA+DB-i(DA-CB)\\&=\overline{\langle y,x\rangle}.\end{aligned}$$

- For a regular element

$$\begin{aligned}&x=\{a\}+i\{b\},\\&\langle x,x\rangle=\langle \{a\}+i\{b\},\{a\}+i\{b\}\rangle\\&=\{a\}\{a\}+\{b\}\{b\}+i(\{b\}\{a\}-\{a\}\{b\})\\&=\{a^2+b^2\}.\end{aligned}$$

Therefore, $\langle x,x\rangle\geq 0$ for any $x\in(I_C)_r$. Further,

$$\langle x,x\rangle=0\Leftrightarrow x=0$$

for any $x\in I_C$.

- Since I_C is a consolidate quasilinear space, $(I_C)^{\wedge}=I_C$ and so

$$\begin{aligned}&\|\langle x,y\rangle\|=\sup\{|t|:t\in x,y\}\\&=\sup\{|t|:t\in AC+BD+i(BC-AD)\}\\&=\sup\{|t_1t_3+t_2t_4+i(t_2t_3-t_1t_4)|:t_1\in A,t_2\in B,t_3\in C,t_4\in D\}\\&=\sup\{|(t_1+it_2)(t_3-it_4)|:t_1\in A,t_2\in B,t_3\in C,t_4\in D\}\\&=\sup\{\|\langle\{t_1\}+i\{t_2\},\{t_3\}+i\{t_4\}\rangle\|:t_1\in A,t_2\in B,t_3\in C,t_4\in D\}\\&=\sup\{\|\langle a,b\rangle\|:a=\{t_1\}+i\{t_2\}\in F_x^{\mathbb{I}_C},b=\{t_3\}+i\{t_4\}\in F_y^{\mathbb{I}_C}\}.\end{aligned}$$

- Suppose that $x\leq y$ and $u\leq v$ for

$$\begin{aligned}&x=[\underline{x_r},\overline{x_r}]+i[\underline{x_s},\overline{x_s}]=A+iB, y=[\underline{y_r},\overline{y_r}]+i[\underline{y_s},\overline{y_s}]=C+iD,\\&u=[\underline{u_r},\overline{u_r}]+i[\underline{u_s},\overline{u_s}]=E+iF \text{ and } v=[\underline{v_r},\overline{v_r}]+i[\underline{v_s},\overline{v_s}]=G+iH.\text{ Then}\\&x\leq y\Leftrightarrow A\subseteq C,B\subseteq D\\&\text{and}\\&u\leq v\Leftrightarrow E\subseteq G,F\subseteq H.\end{aligned}$$

By the seventh axiom of inner-product on I_R we say that $AE \subseteq CG$, $BF \subseteq DH$, $BE \subseteq DG$ and $AF \subseteq CH$. Thus,

$$\begin{aligned}\langle x,u\rangle &= \langle A + iB, E + iF\rangle \\ &= AE + BF + i(BE - AF) \\ &\subseteq CG + DH + i(DG - CH) \\ &= \langle y,v\rangle.\end{aligned}$$

- We show that $x \leq y$, if for any $\varepsilon>0$ there exists an element $x^{\varepsilon} = \left[\underline{x_r^{\varepsilon}}, \overline{x_r^{\varepsilon}}\right] + i\left[\underline{x_s^{\varepsilon}}, \overline{x_s^{\varepsilon}}\right] = A + iB \in \mathbb{I}_{\mathbb{C}}$ such that $x \leq y + x_{\varepsilon}$ and $x^{\varepsilon}, x^{\varepsilon} \subseteq S_{\varepsilon}(\theta)$

where $S_{\varepsilon}(\theta)$ is closed ball of C: By the fact that $\langle x^{\varepsilon}, x^{\varepsilon}\rangle \subseteq S_{\varepsilon}(\theta)$ we have that

$$\| \langle x^{\varepsilon}, x^{\varepsilon}\rangle \| = \| \langle A+iB, A+iB\rangle \| = \| AA+BB+i(BA-AB) \| \leq \varepsilon.$$

If it is considered that $\|A\|^2 \leq \varepsilon$ and $\|B\|^2 \leq \varepsilon$, $\|x^{\varepsilon}\| = \|A+iB\| \leq \|A\|^2 + \|B\|^2 \leq 2\varepsilon$. Thereby, we say that $x \leq y$ by the last condition of norm on $\mathbb{I}_{\mathbb{C}}$.

Remark 3.1 It follows that if $(I_C)_r \equiv C$, *i.e.,* $x = \{a\} + i\{b\}$ *for* $a,b \in R$ *then the inner product (27) on* I_C *coincides with the usual inner product on C.*

In fact, from 27 we obtain the classical norm on I_C defined by

$$\| x \| = \| x, x \|^{1/2} = \sup\{|a+ib| : a \in \mathcal{A}, b \in \mathcal{B}\} \tag{28}$$

where $\mathcal{A} = [\underline{x_r}, \overline{x_r}][\underline{x_r}, \overline{x_r}] + [\underline{x_s}, \overline{x_s}][\underline{x_s}, \overline{x_s}]$ and $\mathcal{B} = [\underline{x_s}, \overline{x_s}][\underline{x_r}, \overline{x_r}] - [\underline{x_r}, \overline{x_r}][\underline{x_s}, \overline{x_s}]$.

Now let us show that the space I_C is Hilbert space, i.e., I_C is complete with the norm defined by (28).

Lemma 3.1 The inequality

$$h(\alpha x, \alpha y) = |\alpha| h(x,y) \tag{29}$$

holds for every element $x = [\underline{x}, \overline{x}], y = [\underline{y}, \overline{y}] \in \mathbb{I}_{\mathbb{R}}, \alpha \in \mathbb{K}$.

Proof. Let us consider the set of pairs $(a_1^{(\varepsilon)}, a_2^{(\varepsilon)})$ providing

$$x \leq y + a_1^{(\varepsilon)},\ y \leq x + a_2^{(\varepsilon)} \text{and} a_i^{(\varepsilon)} \leq \varepsilon, i = 1,2 \tag{30}$$

and the set of this pairs's norms $\left(\|a_1^{(\varepsilon)}\|, \|a_2^{(\varepsilon)}\|\right)$ Then we can write

$$\begin{aligned}&\inf_{\varepsilon>0}\left\{\max\left\{\|a_1^{(\varepsilon)}\|, \|a_2^{(\varepsilon)}\|\right\}\right\} = h(x,y) \\ &= \inf\left\{\varepsilon \geq 0 : x \leq y + a_1^{(\varepsilon)}, y \leq x + a_2^{(\varepsilon)} \text{and} \|a_i^{(\varepsilon)}\| \leq \varepsilon, i = 1,2\right\}.\end{aligned}$$

Now let us take into account the set of pairs $\left(\alpha a_1^{(\varepsilon)}, \alpha a_2^{(\varepsilon)}\right)$ providing the relations

$$\alpha x \leq \alpha y + \alpha a_1^{(\varepsilon)}, \alpha y \leq \alpha x + \alpha a_2^{(\varepsilon)} \tag{31}$$

In general, let $\left\{\left(\|b_1^{(\varepsilon)}\|, \|b_2^{(\varepsilon)}\|\right)\right\}_{\varepsilon}$ be a set providing the properties

$$\alpha x \leq \alpha y + \alpha b_1^{(\varepsilon)}, \alpha y \leq \alpha x + \alpha b_2^{(\varepsilon)} \text{ with } \alpha \neq 0 \tag{32}$$

We obtain

$$x \leq y + \frac{b_1^{(\varepsilon)}}{\alpha},\ \ y \leq x + \frac{b_2^{(\varepsilon)}}{\alpha}$$

by using (11) and (12) from QLS axioms in (32). According to this, the pairs $\left(\frac{b_1^{(\varepsilon)}}{\alpha}, \frac{b_2^{(\varepsilon)}}{\alpha}\right)$ is element of the set of pairs $\left(a_1^{(\varepsilon)}, a_2^{(\varepsilon)}\right)$. Then there exists an element $\left(a_{1_0}^{(\varepsilon)}, a_{2_0}^{(\varepsilon)}\right) \in \left\{\left(a_1^{(\varepsilon)}, a_2^{(\varepsilon)}\right)\right\}_{\varepsilon}$ such that $\left(\frac{b_1^{(\varepsilon)}}{\alpha}, \frac{b_2^{(\varepsilon)}}{\alpha}\right) = \left(a_{1_0}^{(\varepsilon)}, a_{2_0}^{(\varepsilon)}\right)$. Thus $\frac{b_1^{(\varepsilon)}}{\alpha} = a_{1_0}^{(\varepsilon)}$ and $\frac{b_2^{(\varepsilon)}}{\alpha} = a_{2_0}^{(\varepsilon)}$. And then we write $b_1^{(\varepsilon)} = \alpha a_{1_0}^{(\varepsilon)}$ and $b_2^{(\varepsilon)} = \alpha a_{2_0}^{(\varepsilon)}$.

So, we say that

$$\left\{\left(b_1^{(\varepsilon)}, b_2^{(\varepsilon)}\right)\right\} \subseteq \left\{\left(\alpha a_1^{(\varepsilon)}, \alpha a_2^{(\varepsilon)}\right)\right\} \tag{33}$$

Also, it is obvious that

$$\left\{\left(\alpha a_1^{(\varepsilon)}, \alpha a_2^{(\varepsilon)}\right)\right\} \subseteq \left\{\left(b_1^{(\varepsilon)}, b_2^{(\varepsilon)}\right)\right\} \tag{34}$$

taking into account (31) and (32).

By (33) and (34), we say that the set $\left\{\left(\alpha a_1^{(\varepsilon)}, \alpha a_2^{(\varepsilon)}\right)\right\}$ with the set $\left\{\left(b_1^{(\varepsilon)}, b_2^{(\varepsilon)}\right)\right\}$ providing properties in (31) and (32) is same.And also, since

$$\left(\left\|\alpha a_1^{(\varepsilon)}\right\|, \left\|\alpha a_2^{(\varepsilon)}\right\|\right) = \left(|\alpha|\left\|a_1^{(\varepsilon)}\right\|, |\alpha|\left\|a_2^{(\varepsilon)}\right\|\right) = |\alpha|\left(\left\|a_1^{(\varepsilon)}\right\|, \left\|a_2^{(\varepsilon)}\right\|\right),$$

we have

$$\max\left\{|\alpha|\left\|a_1^{(\varepsilon)}\right\|, |\alpha|\left\|a_2^{(\varepsilon)}\right\|\right\} = |\alpha| \max\left\{\left\|a_1^{(\varepsilon)}\right\|, \left\|a_2^{(\varepsilon)}\right\|\right\}$$

and

$$\inf_{\varepsilon>0}\left\{|\alpha| \max\left\{\left\|a_1^{(\varepsilon)}\right\|, \left\|a_2^{(\varepsilon)}\right\|\right\}\right\} = |\alpha| \inf_{\varepsilon>0}\left\{\max\left\{\left\|a_1^{(\varepsilon)}\right\|, \left\|a_2^{(\varepsilon)}\right\|\right\}\right\}.$$

This take also into account, we can say

$$\begin{aligned}
h(\alpha x, \alpha y) &= \inf\left\{\varepsilon \geq 0 : \alpha x \leq \alpha y + b_1^{(\varepsilon)}, \alpha y \leq \alpha x + b_2^{(\varepsilon)} \text{and} \left\|b_i^{(\varepsilon)}\right\| \leq \varepsilon, i = 1,2\right\} \\
&= \inf\left\{\varepsilon \geq 0 : \alpha x \leq \alpha y + \alpha a_1^{(\varepsilon)}, \alpha y \leq \alpha x + \alpha a_2^{(\varepsilon)} \text{and} \left\|\alpha a_i^{(\varepsilon)}\right\| \leq \varepsilon, i = 1,2\right\} \\
&= |\alpha| \inf\left\{\varepsilon \geq 0 : x^\circ y + a_1^{(\varepsilon)}, y^\circ x + a_2^{(\varepsilon)} \text{and} \left\|a_i^{(\varepsilon)}\right\| \leq \varepsilon, i = 1,2\right\} \\
&= |\alpha| h(x, y).
\end{aligned}$$

Theorem 3.3 *I_C is a complete quasilinear space, hence, I_C is a Hilbert quasilinear space.*

Proof. Let $x^{(n)}$ be a Cauchy sequence in I_C, where $x^{(n)} = \left[\underline{x_r^{(n)}}, \overline{x_r^{(n)}}\right] + i\left[\underline{x_s^{(n)}}, \overline{x_s^{(n)}}\right]$. Then, taking into account Lemma 3.1 and Proposition 2.1-ii), for every ε>0 there exists an $n_0 \in N$ such that

$$\begin{aligned}
h(x^{(n)}, x^{(m)}) &= h\left(\left(\left[\underline{x_r^{(n)}}, \overline{x_r^{(n)}}\right] + i\left[\underline{x_s^{(n)}}, \overline{x_s^{(n)}}\right]\right), \left(\left[\underline{x_r^{(m)}}, \overline{x_r^{(m)}}\right] + i\left[\underline{x_s^{(m)}}, \overline{x_s^{(m)}}\right]\right)\right) \\
&\leq h\left(\left[\underline{x_r^{(n)}}, \overline{x_r^{(n)}}\right], \left[\underline{x_r^{(m)}}, \overline{x_r^{(m)}}\right]\right) + h\left(i\left[\underline{x_s^{(n)}}, \overline{x_s^{(n)}}\right], i\left[\underline{x_s^{(m)}}, \overline{x_s^{(m)}}\right]\right) \\
&\leq h\left(\left[\underline{x_r^{(n)}}, \overline{x_r^{(n)}}\right], \left[\underline{x_r^{(m)}}, \overline{x_r^{(m)}}\right]\right) + h\left(\left[\underline{x_s^{(n)}}, \overline{x_s^{(n)}}\right], \left[\underline{x_s^{(m)}}, \overline{x_s^{(m)}}\right]\right) \\
&< \varepsilon, (n, m > n_0)
\end{aligned}$$

Therefore we say that

$$h\left(\left[\underline{x_r^{(n)}}, \overline{x_r^{(n)}}\right], \left[\underline{x_r^{(m)}}, \overline{x_r^{(m)}}\right]\right) < \varepsilon$$

and

$$h\left(\left[\underline{x_s^{(n)}}, \overline{x_s^{(n)}}\right], \left[\underline{x_s^{(m)}}, \overline{x_s^{(m)}}\right]\right) < \varepsilon.$$

These show that the sequences $(x_r^{(n)})$ and $(x_s^{(n)})$ are Cauchy sequence in I_R. Since I_R is a complete metric space,

$$\left(x_r^{(n)}\right) \to \left[\underline{a}, \overline{a}\right], n \to \infty$$

and

$$\left(x_s^{(n)}\right) \to \left[\underline{b}, \overline{b}\right], n \to \infty.$$

Using these limits, we define

$$x = [\underline{a}, \overline{a}] + i[\underline{b}, \overline{b}].$$

From Lemma 3.1 and Proposition 2.1-(ii), we have

$$\begin{aligned} h\left(x^{(n)}, x\right) &= h\left(\left[\underline{x_r^{(n)}}, \overline{x_r^{(n)}}\right] + i\left[\underline{x_s^{(n)}}, \overline{x_s^{(n)}}\right], [\underline{a}, \overline{a}] + i[\underline{b}, \overline{b}]\right) \\ &\le h\left(\left(\left[\underline{x_r^{(n)}}, \overline{x_r^{(n)}}\right], [\underline{a}, \overline{a}]\right) + h\left(i\left[\underline{x_s^{(n)}}, \overline{x_s^{(n)}}\right], i[\underline{b}, \overline{b}]\right)\right) \\ &\le \underbrace{h\left(\left[\underline{x_r^{(n)}}, \overline{x_r^{(n)}}\right], [\underline{a}, \overline{a}]\right)}_{\to 0} + \underbrace{h\left(\left[\underline{x_s^{(n)}}, \overline{x_s^{(n)}}\right], [\underline{b}, \overline{b}]\right)}_{\to 0} \end{aligned}$$

Hence

$$\left[\underline{x_r^{(n)}}, \overline{x_r^{(n)}}\right] + i\left[\underline{x_s^{(n)}}, \overline{x_s^{(n)}}\right] \to [\underline{a}, \overline{a}] + i[\underline{b}, \overline{b}].$$

This shows that x is the limit of $(x^{(n)})$ and proves completeness of I_C because $(x^{(n)})$ was an arbitrary Cauchy sequence.

6. An Application: Representations of some non-deterministic EEG Signals

Sometimes, frequency and time components of a signal are not precisely known. But, we can precisely determine their upper and lower bounds. Further their samples are also not known and however we can restrict its frequency and time to two interval components. Any model including such states can be represented by an interval signals. These or similar situations occurs also in EEG signal processing [12,13]. Processing of this kind circumstances need more extended mathematical analysis than classical analysis and algebra. Consider a discrete-time signal from an EEG detector [6], for more information about the signal processing) and assume we know that the signal is a sinusoidal function sampled at 4 samples per period, for $n \in N$

$$x_n = n + i\left[-1, \sin\frac{\pi n}{2}\right] \text{and } x_n = 0, \text{ for } n \in \mathrm{Z}.$$

That is the output is the discrete-time interval signal:

$$x = (x_n) = (\ldots 0, 0, i[-1, 0], 1 + i\left[-1, \sin\frac{\pi}{2}\right], 2 + i[-1, \sin\pi], 3 + i\left[-1, \sin\frac{3\pi}{2}\right], \ldots)$$

where $i = \sqrt{-1}$,complex unit. This produces a bounded uncertainty in the output. The time component of the signal is classical but the frequency component includes a bounded uncertainty. Because, for example, the frequency of the signal at time 2 may be zero or π or any value at the interval [-1,sinπ]. Processing of this kind signals produces many difficulties since they have no mature mathematical foundations in contrast to the classical signal processing. Suppose that this output is an impulse response of a system. In other words, let us assume it is a filter and let us try to determine which system produce this output:

This kind system is a mapping T from a discrete-time signals to the discrete-time interval-valued periodic signals since an impulse is a classical unit signal. A discrete-time signal f is a two-sided sequence $(\ldots, f_{-1}, f_0, f_1, \ldots)$ and its output under the mapping T is in the from $Tf=z$. The impulse is the Kronecker delta sequence δ such that

$$\delta_n = \begin{cases} 1 \text{, for} n = 0 \\ 0 \text{, otherwise} \end{cases} \qquad n \in \mathbb{Z}$$

or

$$\delta = (\ldots, 0, \underset{0.th \text{ position}}{1}, 0, \ldots).$$

Hence we should solve the equation $T\delta = x$ in order to determine the system. Now we asserts that T is in a complex-interval infinite matrix form: Indeed, if

$$T=\begin{bmatrix} \cdots 0 & \vdots & \vdots & \vdots & \\ \cdots 0 & 0 & 0 & 0 & \cdots \\ \cdots 0 & i[-1,0] & 0 & 0 & \cdots \\ \cdots 0 & 0 & 1+i[-1,\sin\frac{\pi}{2}] & 0 & \cdots \\ \cdots 0 & 0 & 0 & 2+i[-1,\sin\pi] & \cdots \\ \cdots 0 & 0 & 0 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix},$$

the matrix multiplication with impulse gives

$$T(\delta)=(\ldots 0,0,i[-1,0],1+i\left[-1,\sin\frac{\pi}{2}\right],2+i[-1,\sin\pi],3+i\left[-1,\sin\frac{3\pi}{2}\right],\ldots)=x.$$

We call this kind systems as a quasilinear systems and many interval matrix systems represents mentioned uncertainties. Their investigation needs a lot of new concepts and technics.

Conclusion

We call this kind systems as a quasilinear systems and many interval matrix systems represents mentioned uncertainties. Their investigation needs a lot of new concepts and technics.

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