Journal of Biostatistics and Biometric Applications
Volume 3 | Issue 3
ISSN: 2455-765X

Research Article Open Access

Neutrosophic Modules

Olgun N, and Khatib A*

Gaziantep University, Faculty of Science and Art, Department of Mathematics, Gaziantep, Turkey

***Corresponding author:** Khatib A, Gaziantep University, Faculty of Science and Art, Department of Mathematics, Gaziantep, Turkey, Tel: +905535518350, E-mail: kollnaar5@gmail.com

Citation: Olgun N, Khatib A (2018) Neutrosophic Modules. J Biostat Biometric App 3(3): 306

Received Date: February 5, 2018 **Accepted Date**: July 20, 2018 **Published Date:** July 23, 2018

Abstract

The objective of this thesis is to study neutrosophic $R-$ module . Some basic features and definitions of the classical $R-$ module are expanded. It is clear that every neutrosophic $R-$ module over a neutrosophic ring is a $R-$ module. Also, it is shown that an element of a neutrosophic $R-$ module over a neutrosophic ring can be infinitely conveyed as a linear combination of some elements of the neutrosophic $R-$ module. Neutrosophic quotient $R-$ module and neutrosophic $R-$ module homomorphism are also covered.

Keywords: Neutrosophic Ring; Neutrosophic Vector Space; Left Neutrosophic $R-$ module; Right Neutrosophic $R-$ module; Neutrosophic $R-$ module Homomorphism

Introductıon

In our life there is three kinds of logic. the first is classical logic which is gives the form "true or false, 0 or 1" to the values.The second is fuzzy logic was first advanced by Dr. Lotfi Zadeh in 1965s. It recognize more than true and false values which are considered simple. With fuzzy logic, propositions can be represented with degrees of truth and falseness. And the third is neutrosophic logic that is an extending fuzzy logic which includes indeterminacy I.

Since we live in a world filled in indeterminacy, the Neutrosophic found their method into modern research. We can introduce the Neutrosophic Measure and as a result the Neutrosophic Integral and Neutrosophic Probability in several methods, because there are different kinds of indeterminacies, depending on the problem and issue we have to fix. Indeterminacy is distinguished from randomness. Indeterminacy can be caused by physical space materials and type of construction, by items involved in the space, or by other factors. Space objects and structures are the main causes of indeterminacy and other factors can be considerable .

Florentin Smarandanche defined the idea of neutrosophy as a new type of philosophy in 1980. After he found the approach of neutrosophic logic and neutrosophic set where we have a percentage of truth in a subset T and the same of falsity of the subset F, and a percentage of indeterminancy in the subset I for every structure in neutrosophic logic where T , I, F are subset of $\left[{}^{-}0,1^{+}\right]$. Therefore this neutrosophic logic is called en extension of fuzzy logic especially to intuitionistic fuzzy logic.

For more explanation, we can give this simple example: if we say "the weather is hot today". In the classical logic we will say "yes or not , true or false". However, in fuzzy logic we can say "it is 70% and 30% cold" . On the other hand, in neutrosophic logic we can say "It is 60-70% hot, 25-35% cold, and 10% indeterminate".

In deed neutrosophic sets is the extension of classical sets, neutrosophic groups, neutrosophic ring, neutrosophic fields, neutrosophic vector spaces … etc. In the same way neutrosophic $R-$ module is the generalization of classical $R-$ module.

Using the idea of neutrosophic logic, Vasantha Kanadasamy and Florentin Samarandanche studied neutrosophic algebraic structures by using an indeterminate element I in the algebraic structure and then combine (I) with each element of the structure with respect to corresponding binary operation.

The indeterminate element I is in order that if * is ordinary multiplication the multiplication of many (I) is (I) itself and the inverse I^{-1} is not defined and hence is not found. If we have * as an ordinary addition, then the addition of many (I) is (I) itself. They call it neutrosophic element and the generated algebraic structure, is then termed as neutrosophic algebraic structure.

In 1995, Florentin Smarandache introduced the "neutrosophic set theory" to process the indeterminate and inconsistent information which found generally in real cases.

In 2015 Salama introduced the concept of Neutrosophic Crisp set Theory to portray any event by a triple crisp structure. Moreover the work of Salama *et al.* formed a starting point to construct new types of neutrosophic mathematics and computer sciences. Hence, Neutrosophic set theory turned out to be a generalization of both the classical and fuzzy counterparts.

Neutrosophic logic has a broad applications in science, medicine, economics, chemistry, law etc. Therefore, neutrosophic structures are very significant and a broad area of study.

In addition to the introduction, this thesis contains in its second chapter suitable definitions and revision. Moreover , in the third chapter, the modules have been defined through the neutrosophic logic, some examples, theories, and proofs of its legitimacy and validity have been mentioned. Finally, in the fourth chapter, the homomorphisms has been studied with mentioning the theories and examples that support and confirm them.

Fuzzy Set

If we have U as an initial universe and if we take FU as a non-empty set in U. A fuzzy set FU is defined as a set of arranged pairs $\{(f,\mu_{FU}(f))\}$where $f \in U$, the membership function $\mu_{FU}: U \to [0,1]$ of FU and $\mu_{FU}(f) \in [0,1]$ is the degree of membership function of element f in fuzzy set FU for any $f \in U$ [1].

Example: Consider the universe of discourse $U = \{3,5,6,7,9,10\}$. Then a fuzzy set holding the idea of 'large number' can be explained as $A = \{(3,0),(5,0.1),(6,0.2)(7,0.3)(9,0.8)(10,1)\}$ With the considered universe, the numbers 3 is not 'large numbers' ,so the membership degrees equal 0. Numbers 5–9 partially belong to the idea 'large number' with a membership degree of 0.1,0.2 0.3,0.5 and 0.8. then number 10 is the largest number with a full membership degree.

Neutrosophic Numbers

The neutrosophic numbers formed as $x(I) = y + zI \mid y, z \in$ or C , and y is defined as the determinate part on $x(I)$ and zI is defined as the indeterminate part of $x(I)$, with $z_1 I + z_2 I = (z_1 + z_2) I$. If both y ,z are real numbers, then $x(I) = y + zI$ is defined as a neutrosophic real number. If y,z or both are complex numbers, then $x(I) = y + zI$ is defined as a neutrosophic complex number [2].

Example: Let $x(I) = 5 + \sqrt{2}I$ be a real neutrosophic number which has 5 athe determinate part on $x(I)$ and $\sqrt{2}I$ the indeterminate part on $x(I)$.

Similarly let $z(I) = \sqrt{-1}I$ be a complex neutrosophic number which has 0 as the determinate part on $z(I)$ and $\sqrt{-1}I$ the indeterminate part on $z(I)$.

Neutrosophic Sets

If we have U as an initial universe and if we take PH as a subset of U . PH is defined as a neutrosophic set if it was an element $p \in U$ goes back to PH in the following form:

$$p = \left(T_{PH}(p), I_{PH}(p), F_{PH}(p)\right) \in U$$

i) $T_{PH}(P)$ is t % true in PH where $t \in T$.

ii) $I_{PH}(P)$ is i % true in PH where $i \in I$.

iii) $F_{PH}(P)$ is f % true in PH where $f \in F$.

The result $t+i+f=1$ refers that it is possible like in the situation of classical and fuzzy logics. Also the result $t+i+f<1$ refers that it is possible like in the situation of intuitionistic logic and the result $t+i+f>1$ refers that it is possible like in the situation of paraconsistent logic [3].

Example: The probability of a patient to pass his surgery is "60% true" according to his doctor in the hospital, "25 or 30-35% false" according to his weak immunity, and "15 or 20% indeterminate" due to equipment in the hospital.

The Complement Of a Neutrosophic Set: If we have U as an initial universe and if we take PH as a neutrosophic subset of U , then the complement of PH is indicated as $(PH)^c$ and is defined as the following way:

$$T_{(PH)^c}(k) = F_{(PH)^c}(k)$$

$$I_{(PH)^c}(k) = \{1^+\} - I_{(PH)^c}(k)$$

$$F_{(PH)^c}(k) = T_{(PH)^c}(k)$$

$$\forall k \in U, T(k)_{PH}, I(k)_{PH}, F(k)_{PH} \in [0,1] \quad (4)$$

The Containment Of Two Neutrosophic Sets: If we have U as an initial universe and if we take PH, SH as two neutrosophic subsets of U, we say PH is contained in SH and indicated by $PH \subseteq SH$, precisely when following holds:

$$T_{SH} \leq T_{PH}$$

$$I_{SH} \leq I_{PH}$$

$$F_{SH} \geq F_{PH}$$

$$\forall k \in U, T(k)_{PH}, I(k)_{PH}, F(k)_{PH} \in [0,1]$$

$$\forall k \in U, T(k)_{SH}, I(k)_{SH}, F(k)_{SH} \in [0,1] \quad (4)$$

The Union Of Two Neutrosophic Sets: If we have U as an initial universe and PH, SH are two subsets of U, the union

of these neutrosophic sets PH and SH will be a neutrosophic set PS, and we write $PS = PH \cup SH$, if and only if the conditions holds:

$$T_{PS}(k) = \max\left(T_{PH}(k), T_{SH}(k)\right)$$

$$I_{PS}(k) = \max\left(I_{PH}(k), I_{SH}(k)\right)$$

$$F_{PS}(k) \quad \min\left(F_{PH}(k), F_{SH}(k)\right)$$

$$\forall k \in U, T(k)_{PH}, I(k)_{PH}, F(k)_{PH} \in [0,1]$$

$$\forall k \in U, T(k)_{SH}, I(k)_{SH}, F(k)_{SH} \in [0,1] \quad (4)$$

The Intersection Of Two Neutrosophic Sets: If we have U as an initial universe and if we take PH , SH as two subsets of U, the intersection of this neutrosophic sets PH and SH will be neutrosophic set PS and we write $PS = PH \cap SH$, if and only if the conditions holds:

$$T_{PS}(k) = \min\left(T_{PH}(k), T_{SH}(k)\right)$$

$$I_{SP}(k) = \min\left(I_{PH}(k), I_{SH}(k)\right)$$

$$F_{PS}(k) \quad \max\left(F_{PH}(k), F_{SH}(k)\right)$$

$$\forall k \in U, T(k)_{PH}, I(k)_{PH}, F(k)_{PH} \in [0,1]$$

$$\forall k \in U, T(k)_{SH}, I(k)_{SH}, F(k)_{SH} \in [0,1] \qquad (4)$$

The Difference of Two Neutrosophic Sets: If we have U as an initial universe and if we take PH , SH as two subsetsof U , the difference of these neutrosophic sets PH and SH will be neutrosophic set

PS and we write $PS = PH - SH$, if and only if the conditions holds:

$$T_{PS}(k) = \min\left(T_{PH}(k), F_{SH}(k)\right)$$

$$I_{PS}(k) = \min\left(I_{PH}(k), 1 - I_{SH}(k)\right)$$

$$F_{PS}(k) = \min\left(F_{PH}(k), T_{SH}(k)\right)$$

$$\forall k \in U, T(k)_{PH}, I(k)_{PH}, F(k)_{PH} \in [0,1]$$

$$\forall k \in U, T(k)_{SH}, I(k)_{SH}, F(k)_{SH} \in [0,1] \qquad (4)$$

Example: If we have U as an initial universe and if we take PH , SH as two neutrosophic sets of U like the following :

$$PH = \left\{\left\langle k_1(0.3, 0.5, 0.6)\right\rangle, \left\langle k_2(0.3, 0.2, 0.4)\right\rangle, \left\langle k_3(0.8, 0.7, 0.2)\right\rangle\right\}$$

$$SH = \left\{\left\langle k_1(0.6, 0.1, 0.2)\right\rangle, \left\langle k_2(0.3, 0.2, 0.6)\right\rangle, \left\langle k_3(0.4, 0.1, 0.5)\right\rangle\right\}$$

where $k_1, k_2, k_3 \in [0,1]$

then:

1) $$(PH)^c = \left\{\left\langle k_1(0.6, 0.5, 0.3)\right\rangle, \left\langle k_2(0.4, 0.8, 0.3)\right\rangle, \left\langle k_3(0.2, 0.3, 0.8)\right\rangle\right\}$$

2) $(SH)^c = \{\langle k_1(0.2,0.9,0.6)\rangle, \langle k_2(0.6,0.8,0.3)\rangle, \langle k_3(0.5,0.9,0.4)\rangle\}$

3) $PH \cup SH = \{\langle k_1(0.6,0.5,0.2)\rangle, \langle k_2(0.3,0.2,0.4)\rangle, \langle k_3(0.8,0.7,0.2)\rangle\}$

4) $PH \cap SH = \{\langle k_1(0.3,0.1,0.2)\rangle, \langle k_2(0.3,0.2,0.6)\rangle, \langle k_3(0.4,0.1,0.5)\rangle\}$

5) $PH - SH = \{\langle k_1(0.2,0.5,0.6)\rangle, \langle k_2(0.3,0.2,0.3)\rangle, \langle k_3(0.5,0.7,0.2)\rangle\}$

Neutrosophic Groups

If we have $(G,\lozenge)$ as a group , $PHG = (G(I),\lozenge)$ is defined as a neutrosophic group which is formed by *I* and *G* with $\lozenge$ [5].

• we notice that *PHG* is a commutative neutrosophic group if $g_1 \lozenge g_2 = g_2 \lozenge g_1$ for all $g_1, g_2 \in PHG$.

Theorem: If we have *PHG* as a neutrosophic group ,then *PHG* always is not a group but it must have a group[5].

Example 1: Under multiplication modulo 3 we have $Z_3 | \{0\} = \{1,2\}$ is a group but $PHZ_3 | \{0\} = \{1,2,I,2I\}$ is not a group .

Example 2: The groups $(PHZ,+)$,$(PHQ,+)$,$(PHR,+)$ and $(PHC,+)$ are neutrosophic groups with (+) [6].

Example 3: The groups $(PH\{Q-\{0\}\},\cdot)$,$(PH\{R-\{0\}\},\cdot)$ and $PH(\{C-\{0\}\},\cdot)$ are neutrosophic groups with $(\cdot)$.

Example 4: If we have $PHG = \{e, g_1, g_2, g_3, I, g_1 I, g_2 I, g_3 I\}$ be a set where $g_1^2 = g_2^2 = g_3^2 = e$ and $g_1g_2=g_2g_1=g_3$,$g_3g_2=g_2g_3=g_1$,$g_1g_3=g_3g_1=g_2$, then under multiplication *PHG* is a commutative ,for {e, g_1, g_2, g_3} we have a Klein group [6].

Example 5: If we have the group $PHG = \left\{\begin{bmatrix} m & n \\ k & p \end{bmatrix} : m,n,k,p \in \{1,4,7,I,2I\}\right\}$ with matrix multiplication modulo 3 , it easy to find *PHG* is a non-commutative neutrosophic group.

Neutrosophic Subgroups: If we have *PHG* as a neutrosophic group and if we take *SHG* as a subset in *PHG* , we define *SHG* as a neutrosophic subgroup precisely when:

1) $SHG \neq \varnothing$.
2) *SHG* itself is a neutrosophic group.
3) *SHG* must has a subset which is a group[5].

Example: If we have The neutrosophic group $PHG = \{e, g_1, g_2, g_3, I, g_1 I, g_2 I, g_3 I\}$ then the subgroups $SHG = \{e, g_1, I, g_2 I\}$, $NHG = \{e, g_2, I, g_2 I\}$ and $KHG = \{e, g_3, I, g_3 I\}$ are neutrosophic subgroups of *PHG* [6].

The Order Of Neutrosophic Group

If we have *PHG* as a neutrosophic group. The order of *PHG* is the cardinal number $|PHG|$. *PHG* is known as finite (resp. infinite) if $|PHG|$ is finite (resp. infinite) [5].

The Neutrosophic Element And Free Element

If we have *PHG* as a neutrosophic group, and we have an element $k \in PHG$ which is called a neutrosophic element if is $t \in Z^+$ it makes $k^t = I$ exist, and if $t \in Z^+$ does not exist so *k* is called a neutrosophic free element [5].

Example: If we have *PHG* = {1, 2, 5, I, 3I, 9I, 11I} under multiplication modulo 10 ,then:

- $|PHG| = 7$
- (I) ,(3I), (9I), (11I), neutrosophic elements.
- 1,2,5 a free neutrosophic elements.

Neutrosophic Rings

If we have $(R,+,\cdot)$ as any ring and *PH(R)* neutrosophic set formed by *R* and *I* , we define the triple $PH(R) = (R(I),+,\cdot)$ as a neutrosophic ring [5].

- If $r \cdot s = s \cdot r$ for any $r,s \in PH(R)$ we notice that it is a commutative neutrosophic ring.
- It is obvious $R \subseteq PH(R)$.

Example: $PHZ = \{t_1 + t_2 I \mid t_1, t_2 \in Z\}$ is a ring known as the commutative neutrosophic ring of integers[5].

Example: If we have $(Q,+,\cdot)$ as a ring of rational ,then $PHQ = \{q_1 + q_2 I \mid q_1, q_2 \in Q\}$ is the ring known as the commutative neutrosophic ring of rational numbers [5].

Example: If we have $(R,+,\cdot)$ as a ring of real numbers ,then $PHR = \{r_1 + r_2 I \mid r_1, r_2 \in R\}$ is the neutrosophic ring known as the commutative neutrosophic ring of real numbers [5].

Example: If we have $PH(R) = \left\{\begin{bmatrix} q_1 & q_2 \\ q_3 & q_4 \end{bmatrix} : q_1, q_2, q_3, q_4 \in PHQ\right\}$, then *PH(R)* is a non-commutative with matrix addition and multiplication.

Theorem: Every neutrosophic ring *PH(R)* is a ring *R* [5].

The Neutrosophic Ring of Characteristic

If we have *PH(R)* as a neutrosophic ring, and if there is at least t like $tr = 0, t \in Z^+$ for all $r \in PH(R)$, then *PH(R)* is said to have characteristic t . It is possible only if $t = 0$, then *PH(R)* is said to have characteristic zero. .

Example: We have the neutrosophic ring $PH(Q) = \langle Q \cup I \rangle$ it's a neutrosophic ring of characteristic zero .

Neutrosophic Subrings

If we have *PH(R)* as a neutrosophic ring and if we take *SH(R)* as a subset in *PH(R)* , we define *SH(R)* as a neutrosophic subring precisely when:

1) $SH(R) \neq \varnothing$
2) *SH(R)* itself is a neutrosophic ring.
3) *SH(R)* must has a proper subset which is a ring [5].

Example: The set *SHE* of even neutrosophic integers is a commutative subring of integers neutrosophic ring *PHZ*.

Neutrosophic Ideals

If we have *PH(R)* as a neutrosophic ring . A neutrosophic subring *SH(J)* of *PH(R)* is said to be an ideal of *PH(R)* Precisely when : $\forall r \in PH(R)$, $\forall j \in SH(J) \Rightarrow rj, jr \in SH(J)$.

Example: *SH(2Z)* is a neutrosophic ideal of *PH(Z)* .

Neutrosophic Fields

If we have $(F,+,\cdot)$ as a field and if we take *PH(F)* as a set formed by F with I , we will define the form $PH(F) = (F(I),+,\cdot)$ as a neutrosophic field [5].

- For $\alpha \in F$ then $\alpha \cdot I = \alpha I$, $0 \in PHF$ is formed by 0 + 0*I* , 0*I* =0 and $1 \in PHF$ is formed by 1+0*I* in *PHF* .

Example 1: *PH(Q)* is the neutrosophic field of rationale numbers which formed by rationale numbers and *I* [5].

Example 2: *PH(R)* is the neutrosophic field of real numbers which formed by real numbers and *I* [5].

Example 3: *PH(C)* is the neutrosophic field of complex numbers which formed by complex numbers and *I* [5].

Neutrosophic Subfields: If we have *PH(F)* as a neutrosophic field and if we take S*P(F)* as a subset of of *PH(F)*, we will define S*P(F)* as a neutrosophic subfield precisely when:

1) $SP(F) \neq \varnothing$.
2) S*P(F)* itself is a neutrosophic field.
3) S*P(F)* must has a subset which is a field [5].

Example: *PH(R)* is neutrosophic subfield of *PH(C)* .

Neutrosophic Vector Spaces

If we have $(V,+,\cdot)$ as a vector space over F and if we take the set *PH(V)* which formed by V and *I* , we will define the form

$(PH(V),+,\cdot)$ as a weak neutrosophic vector space (WNV) over F. However, if F is a $PH(F)$ then $(PH(V),+,\cdot)$ is defined as a strong neutrosophic vector space (SNV) over *PH(F)* [3].

Example: We can look at *PH(R)* through two views it's a (WNV) over Q. Moreover, it's a (SNV) over *PH(Q)* [3].

Neutrosophic Subspace: If we have *PH(V)* as a (*WNV*) over F and if we take (*SHV*) as a subset of *PH(V)*, we will define (*SHV*) as a (WNV) subspace of *PH(V)* precisely when:

1) $SH(V) \neq \varnothing$
2) (*SHV*) itself a weak neutrosophic vector space o ver F.
3) (*SHV*) must has a subset which is a vector space [3].

• Similarly for (SNV) subspace .

Example 1: If we have *PH(V)* as (WNV) or (SNV) , then *PH(V)* is a subspace of itself and known as a trivial (WNV) or (SNV) subspace [3].

Example 2: If we have $PH(V) = PHM_{m\times n} = \{[m_{ij}] : m_{ij} \in PH(R)\}$ as a (SNV) over *PH(R)* . And if we take the subset $SH(V) = SHN_{m\times n} = \{[n_{ij}] : n_{ij} \in PH(R), trace(N) = 0\}$,then (*SHV*) is a (SNV) subspace of *PH(V)* [3].

Neutrosophic Left Module

If we have $({}_RM,+,\cdot)$ as a left $R-$ module over a ring R and we have $PH({}_RM)$ as a neutrosophic non-empty set formed by ${}_RM$ and I, then the triple $PH({}_RM) = ({}_RM(I),+,\cdot)$ is defined as a weak neutrosophic left $R-$ module (WNML) over R.

• If R is *PH(R)*, then $PH({}_{PHR}M)$ is defined as a strong neutrosophic left $R-$ module (SNML) over *PH(R)* [7-12].

Neutrosophic Right $R-$ Module

Similarly the form $PH(M_R) = (M_R(I),+,.)$ is known as weak neutrosophic right $R-$ module (WNML) over a ring R.

• If R is *PH(R)* , then $PH(M_{PHR})$ is defined as a strong neutrosophic right $R-$ module (SNMR) over *PH(R)* .

Notes :

1) If we have R (*PH(R)*) as a commutative ring (neutrosophic ring), then every $PH({}_RM)$ is a $PH(M_R)$ ($PH({}_{PHR}M)$ is a $PH(M_{PHR})$).

2) in general neutrosophic $R-$ module (strong ,weak ,lift and ,right) denoted by $PH(M)$ or (NM).

3) If $\{m,n \in PH(M) : m = a+bI, n = k+tI\}$ where a,b,k and t are elements in M and $\{r \in R(I) : r = p+qI\}$ where P and q are scalars in R we define:

i) $$m+n = (a+bI)+(k+tI)$$
$$= (a+k)+(b+t)I$$

ii) $$rm = (p+qI)(a+bI)$$
$$= pa+(pb+qa+qb)I$$

Example 1: If we have *PH(R)* as a (NR) , and if we take *PH(J)* as a (NI) of *PH(R)* , then :

1) *PH(R)* is a *PH(M)* .

2) *PH(J)* is a neutrosophic $R-$ module under the addition and multiplication of *PH(R)* .

3) *PH(R)/ PH(J)* is a *PH(M)* .

Example 2: We can look at $PH(R^n)$ through two views: the first is a (WNM) over a ring R, and the second is a (SNM) over a neutrosophic ring *PH(R)* .

Example 3: We can look at $PH(M_{m\times n}) = \{[a_{ij}] : a_{ij} \in PH(Q)\}$ through two views: the first is a (WNM) over a ring Q, and the second is a (SNM) over *PH(Q)* .

Theorem 4: If we have a (SNM) then it is a (WNM).

Proof : If we have *PH(M)* as a (SNM) because it is $R \subseteq PH(R)$ for every ring R , then *PH(M)* is a (WNM) over R .

Theorem: If we have *PH(M)* as a commutative group then every weak (strong) neutrosophic $R-$ module is a $R-$ module.

Proof: If we have $m = a + bI, n = k + tI \in PH(M)$ where $a, b, k, t \in M$ and $\alpha = p + qI, \beta = r + sI \in PH(R)$ $: p, q, r, s \in R$ then :

1) $\alpha(m+n) = (p+qI)(a+bI+k+tI)$

$$= pa + pk + (pb + pt + qa + qb + qk + qt)I$$

$$= \alpha m + \alpha n$$

2) $(\alpha+\beta)m = (p+qI+r+sI)(a+bI)$

$$= pa + ra + (qp + rb + sa + sb)I$$

$$= \alpha m + \beta m$$

3) $(\alpha\beta)m = ((p+qI)(r+sI))(a+bI)$

$$= pra + (prb + psa + psb + qra + qrb + qsa + qsb)I$$

$$= \alpha(\beta m)$$

4) For $1 + 0I \in PH(R)$ we have:

$$1m = (1+0I)(a+bI)$$

$$= 1a + 1bI + 0aI + 0bI^2$$

$$= a + bI + 0 + 0 = a + bI = m$$ So that , *PH*(*M*) is a $R-$ module.

Lemma : If we have *PH*(*M*) as a neutrosophic $R-$ module over ring *PH*(*R*) and if we take $m = a + bI, h = k + tI, s = e + fI \in PH(M)$, $\alpha = p + qI \in PH(R)$ then:

1) $m + s = s + h \Rightarrow m = s$

2) $\alpha 0 = 0$

3) $0m = 0$

4) $(-\alpha)m = \alpha(-m) = -(\alpha m)$

Neutrosophic Submodule

If we have *PH*(*M*) as a neutrosophic $R-$ module over a ring *PH*(*R*) and if we take *SH*(*N*) as a subset of , we will define as a strong neutrosophic submodule precisely when:

1) $SH(N) \neq \varnothing$

2) *SH*(*N*) itself is a strong neutrosophic $R-$ module.

3) *SH*(*N*) must has a proper subset which is a $R-$ module.

• Similarly for weak neutrosophic submodule.

Theorem: If we have *PH(M)* as a neutrosophic $R-$ module over a ring *PH(R)* and if we take *SH(N)* as a subset of , we will define as a strong neutrosophic submodule precisely when:

1) $SH(N) \neq \varnothing$
2) $n, n' \in SH(N) \Rightarrow n + n' \in SH(N)$.
3) $n \in SH(N), \alpha \in PH(R) \Rightarrow \alpha n \in SH(N)$.
4) *SH(N)* must has a proper subset which is a $R-$ module.

Corollary : If we have *PH(M)* as a (NM) over a ring *PH(R)* and if we take *SH(N)* as a subset of *PH(M)* ,then *SH(N)* is neutrosophic submodule of *PH(M)* precisely when:

1) $SH(N) \neq \varnothing$
2) $\forall h, t \in SH(N), \forall r, q \in PH(R) \Rightarrow rh + qt \in SH(N)$.
3) *SH(N)* must has a proper subset which is a $R-$ module.

Example 1: If we have *PH(M)* as a (NM) over a ring *PH(R)* ,then *PH(M)* is a neutrosophic submodule known as a trivial neutrosophic submodule.

Example 2: If we have $PH(M) = PH(M_{m\times n}) = \{[a_{ij}] : a_{ij} \in R\}$ as a (NM) over *R* and $SH(N) = SH(B_{m\times n}) = \{[b_{ij}] : b_{ij} \in R, trace(B) = 0\}$, then *SH(N)* is a neutrosophic submodule of *PH(M)* .

Example 3: If we have *PH(M)* = *PH*(R^3) as a (NM) over a *PH(R)* and if we take $SH(N) = \{n = x + yI, n' = z + tI, 0 = 0 + 0I \in PH(M) : x, y, z, t \in M\}$, then *SH(N)* is a neutrosophic submodule of *PH(M)* .

Theorem: If we have *PH(M)* as a (NM) over a ring *PH(R)* and if we take $\{SH(N_i)\}_{i\in A}$ as a set of all neutrosophic submodule of *PH(M)*, then $\cap SH(N)$ is neutrosophic submodule of *PH(M)* .

Proof: It's clearly $\cap SH(N) \neq \varnothing$, then:

1) If we have $k, t \in \cap SH(N) \Rightarrow k - t \in \cap SH(N)$.

2) If we have $h \in SH(N)$ and $\alpha \in PH(R) \Rightarrow \alpha h \in \cap SH(N)$.

Since for $\forall i \in A$ implies $\cap SH(N)$ is neutrosophic submodule of $PH(M)$.

Remark: If we have *PH(M)* as a (NM) over a ring *PH(R)*, and *SH(A)* , *SH(B)* as two different neutrosophic submodules of *PH(M)*. In general, $SH(A) \cup SH(B)$ is not a neutrosophic submodule of *PH(M)* . However, if $SH(A) \subseteq SH(B)$ or $SH(B) \subseteq SH(A)$ true, then $SH(A) \cup SH(B)$ is a neutrosophic submodule of *PH(M)*.

The Sum And Direct Sum Of Two Neutrosophic Submodule

If we have *SH(A)* and *SH(B)* as two neutrosophic submodules of *PH(M)*

over a neutrosophic ring *PH(R)* then:

1) We define the sum of *SH(A)* and *SH(B)* by the set:

$\{n_1 + n_2 : n_1 \in SH(A), n_2 \in SH(B)\}$ and refer by $SH(A) + SH(B)$.

2) *PH(M)* is said to be the direct sum of *SH(A)* and *SH(B)* precisely when:

$\forall m \in PH(M) \Rightarrow m = n_1 + n_2$ where $n_1 \in SH(A)$ and $n_2 \in SH(B)$.

We denoted by $PH(M) = SH(A) \oplus SH(B)$.

Example: If we have *PH(M)* = *PH*(R^3) as a (NM) over a ring *PH(R)* and if we take *SH(A)*, *SH(B)* as two neutrosophic submodules of *PH(M)* like :

$$SH(A) = \{(n_1, 0, 0) : n_1 \in PH(R)\}$$

$$SH(B)=\{(0,n_2,n_3):n_1,n_2\in PH(R)\}$$

Then $PH(M)=SH(A)\oplus SH(B)$.

Lemma: If we have *SH(A)* as a neutrosophic submodule of a neutrosophic $R-$ module *PH(M)* over neutrosophic ring *PH(R)*, then:

1) $SH(A)+SH(A)=SH(A)$

2) $n+SH(A)=PH(A)$, $\forall n\in SH(A)$.

Theorem: If we have *SH(A)* and *SH(B)* as two neutrosophic submodules of *PH(M)* over a neutrosophic ring *PH(R)* , then:

1) $SH(A)+SH(B)$ is a neutrosophic submodule of *PH(M)*.

2) *SH(A)* and *SH(B)* are contained in $SH(A)+SH(B)$.

Proof: 1) Clearly , A+B is a submodules contained in *SH(A) + SH(B)* .

Let $n,k\in SH(A)+SH(B)$ and let $\alpha,\beta\in PH(R)$ Then:

$n=(n_1+n_2I)+(k_1+k_2I)$, $k=(n_3+n_4I)+(k_3+k_4I)$ where

$n_i\in N_1,k_i\in N_2,i=1,2,3,4$

$\alpha=a+bI,\beta=c+dI$ where $a,b,c,d\in R$ .Now,

$$\alpha n+\beta k=\left[(an_1+cn_3)+(an_2+bn_1+cn_4+dn_3+dn_4)I\right]$$

$$+\left[(ak_1+ck_3)+(ak_2+bk_1+ck_4+dk_3+dk_4)I\right]$$

$$\Rightarrow \alpha n+\beta k\in SH(A)+SH(B)$$

Accordingly, *SH*(*A*) + *SH*(*B*) is a neutrosophic submodules of *PH*(*M*) .

2) Clear.

Theorem: If we have *SH*(*A*) and *SH*(*B*) as two neutrosophic submodules of *PH*(*M*) over a neutrosophic ring *PH*(*R*) , then $PH(M)=SH(A)\oplus SH(B)$ if and only if:

1) *PH*(*M*) = *SH*(*A*) + *SH*(*B*)

2) $SH(A)\cap SH(B)=\{0\}$.

Theorem: If we have *SH*(*A*) and *SH*(*B*) as two neutrosophic submodules of *PH*(*M*) over a neutrosophic ring *PH*(*R*) , then $SH(A)\times SH(B)=\{(n_1,n_2):n_1\in SH(A),n_2\in SH(B)\}$ is a neutrosophic $R-$ module over a ring *PH*(*R*) where addition and multiplication are defined by the form:

1) $(m_1,m_2)+(n_1,n_2)=(m_1+n_1,m_2+n_2)$

2) $\alpha(m_1,m_2)=(\alpha m_1,\alpha m_2)$

The Neutrosophic Quotient $R-$ module

If we have *SH(N)* as a neutrosophic submodule of a neutrosophic $R-$ module *PH(M)* over a ring *PH(R)* , the quotient *PH(M)/SH(N)* is defined as $\{n+SH(N):n\in PH(M)\}$ which can be made a neutrosophic $R-$ module over a ring *PH(R)* . We can define the addition and multiplication as in the following way:

$$\forall\left(a+SH(N)\right),\left(b+SH(N)\right)\in PH(M)/SH(N) \text{ and } \forall\alpha\in PH(R)$$

$$\left(a+SH(N)\right)+\left(b+SH(N)\right)=(a+b)+SH(N),$$

$$\alpha\left(a+SH(N)\right)=\alpha a+SH(N).$$

The neutrosophic $R-$ module $PH(M)$ / $SH(N)$ over a $PH(R)$ is defined as a neutrosophic quotient $R-$module.

Example: If we have $PH(M)$ as a neutrosophic $R-$ module over a $PH(R)$, then $PH(M)$ / $PH(M)$ is a neutrosophic zero $R-$ module.

Neutrosophic Linear Combination , Neutrosophic Linearly Independent Set And Neutrosophic Linearly Dependent Set

If we have $PH(M)$ as a neutrosophic $R-$ module over a ring $PH(R)$.

$\forall m_1,m_2,...,m_n\in PH(M)$, then:

1) An element $m\in PH(M)$ is known as a linear combination of the $\{\ m_1,m_2,...,m_n\ \}$ if $\{\ m_1,m_2,...,m_n\ \}$ where $\alpha_i\in PH(R)$.

2) If $\alpha_1 m_1+\alpha_2 m_2+...+\alpha_i m_n=0 \Rightarrow \alpha_1=\alpha_2=...=\alpha_i=0$ (all α_i are equal to zero), then, $\{\ m_1,m_2,...,m_n\ \}$ is defined as a linearly independent set.

3) $\{\ m_1,m_2,...,m_n\ \}$ are said to be linearly dependent if $\alpha_1 m_1+\alpha_2 m_2+...+\alpha_i m_n=0$ implies that not all α_i are equal to zero, then $\{\ m_1,m_2,...,m_n\ \}$ is defined as a linearly dependent set.

Theorem: If we have $PH(M)$ as a neutrosophic $R-$ module over a ring $PH(R)$ and if we take $SH(A)$ and $SH(B)$ as two subsets of $PH(M)$ as $SH(A)\subseteq SH(B)$, then:

If $SH(A)$ is linearly dependent as a result $SH(B)$ is linearly dependent.

Corollary: If we have $PH(M)$ as a neutrosophic $R-$ module over a ring $PH(R)$ and if we take $\{\ m_1,m_2,...,m_n\ \}$ as a linearly dependent set in $PH(M)$, then every subset of $\{\ m_1,m_2,...,m_n\ \}$ will be linearly dependent set too.

Theorem: If we have $PH(M)$ as a neutrosophic $R-$ module over a ring $PH(R)$ and if we take $SH(A)$ and $SH(B)$ as two subsets of $PH(M)$ as $SH(A)\subseteq SH(B)$, then:

If $SH(A)$ is linearly independent as a result $SH(B)$ is linearly independent.

Example: If we have $PH(M)$ as a neutrosophic $R-$ module over a ring $PH(R)$, an element $m=8+4I\in PH(M)$ is a linear combination of the elements

$m_1=1+2I, m_2=2+3I\in PH(M)$ since $8+4I=-16(1+2I)+12(2+3I)$.

Theorem: If we have $PH(M)$ as a neutrosophic $R-$ module over a ring $PH(R)$ and if we take $m_1,m_2,...,m_n\in PH(M)$ and $m\in PH(M)$,then we can infinitely explained as a linear combination of the $\{\ m_1,m_2,...,m_n\ \}$.

Proof: if we have $m=m_1\alpha_1+m_2\alpha_2+...+m_n\alpha_n$

where $m=a+bI, m_1=a_1+b_1I, m_2=a_2+b_2I,...m_n=a_n+b_nI\in PH(M)$

and $\alpha_1=x_1+y_1I,\alpha_2=x_2+y_2I,...,\alpha_n=x_n+y_nI\in PH(R)$

Then : $a+bI=(x_1+y_1I)(a_1+b_1I)+(x_2+y_2I)(a_2+b_2I)+...+(x_n+y_nI)(a_n+b_nI)$ from which we have

$$a_1x_1+a_2x_2+...+a_nx_n=a$$

$$b_1x_1+a_1y_1+b_1y_1+b_2x_2+a_2y_2+...+b_nx_n+a_ny_n+b_ny_n=b$$

That means there are many solution of $x_t, y_t : t = 1,2,...,n$. It implies that the $\{ m_1, m_2, ..., m_n \}$ can be infinitely combined to produce.

Notice : If we have *PH*(*M*) as a neutrosophic $R-$ module over a ring *PH*(*R*), and we have $\alpha m = 0$, this does not mean $\alpha = 0 \in PH(R)$ or $m = 0 \in PH(M)$

all the times it is possible to have $m \neq 0$ and $\alpha \neq 0$. For example , if $m = x - xI, m \in PH(M), x \neq 0$ and $\alpha = yI$ where $\alpha \in PH(R)$, $x, y \in R$ we have $\alpha m = yI(x + xI) = yxI - yxI = 0$.

Theorem: If we have *PH*(*M*) as a neutrosophic $R-$ module over a ring

PH(*R*) and $m_1 = x_1 - x_1 I, m_2 = x_2 - x_2 I, ..., m_n = x_n - x_n I \in PH(M)$ where $x_i \neq 0 \in R$, then $\{ m_1, m_2, ..., m_n \}$ is a linearly dependent set.

Proof : If we have $\alpha_1 = p_1 + q_1 I, \alpha_2 = p_2 + q_2 I, ..., \alpha_n = p_n + q_n I \in PH(R)$, then $\alpha_1 m_1 + \alpha_2 m_2 + ... + \alpha_n m_n = 0$

$$\Rightarrow (p_1 + q_1 I)(x_1 + x_1 I) + (p_2 + q_2 I)(x_2 + x_2 I) + ... + (p_n + q_n I)(x_n + x_n I) = 0$$

$$\Rightarrow x_1 p_1 + x_2 p_2 + ... + x_n p_n = 0$$

That means there are many nontrivial solution of $p_i : i = 1,2,...,n$. As a result, $\{ m_1, m_2, ..., m_n \}$ is a linearly dependent set.

Example: If we have *PH*(*M*) = *PH*(R^n) as a neutrosophic $R-$ module over a ring *PH*(*R*) .As an example of a linearly independent set in *PH*(*M*) we can take the set $\left\{ \begin{array}{l} m_1 = (1,0,0,...,0), m_2 (0,1,0,...,0), ..., m_s (0,0,...,1) \\ , m_{s+1} (I,0,0,...,0), m_{s+2} (0,I,0,...,0), ..., m_n (0,0,...,I) \end{array} \right\}$.

Theorem: If we have *PH*(*M*) as a neutrosophic $R-$ module over a ring *PH*(*R*) and if we take *SH*(*N*) as a nonempty subset of *PH*(*M*), we refer to the all linear combinations of *SH*(*N*) by the form $\xi(SH(N))$, then:

1) $\xi(SH(N))$ is a (SN) submodule of *PH*(*M*) having *SH*(*N*) .

2) If we have *SH*(*A*) as a (SN) submodule of *PH*(*M*) having *SH*(*N*) , then $\xi(SH(N)) \subset SH(A)$.

Proof: 1) Clearly, because *SH*(*N*) is a nonempty set, as a result $\xi(SH(N))$ is nonempty . $\forall m = x + yI \in SH(N)$, $\alpha = 1 + 0I$, if we write

$$\alpha m = (1 + 0I)(x + yI)$$

$$= x + yI \in \xi(SH(N))$$

$$\Rightarrow SH(N) \subset \xi(SH(N))$$

Finally, $\forall m, n \in \xi(SH(N))$, $m_i, n_j \in SH(N), \alpha_i, \beta_j \in PH(R)$ then :

$$m = \alpha_1 m_1 + \alpha_2 m_2 + ... + \alpha_n m_n$$
$$n = \beta_1 n_1 + \beta_2 n_2 + ... + \beta_n n_n$$

$\Rightarrow \alpha m + \beta n \in \xi(SH(N))$ For $\alpha, \beta \in PH(R)$. Since *SH*(*N*) is a subset of $\xi(SH(N))$ which is a submodule of *PH*(*M*) having *SH*(*N*), $\xi(SH(N))$ is a neutrosophic submodule of *PH*(*M*) having *SH*(*N*) .

2) As it is in the classical case and it is cancelled .

The Span Of Neutrosophic module

If we have *PH*(*M*) as a neutrosophic $R-$ module over a ring *PH*(*R*) , the neutrosophic submodule $\xi(SH(N))$ of theorem 3.5.5 is defined as the span of *SH*(*N*) and it is referred to as $spanSH(N)$.

• If $PH(M) = spanSH(N)$, then we say *SH*(*N*) a *span PH(M)* .

The Basis Of Neutrosophic module

If we have *PH(M)* as a neutrosophic $R-$ module over a ring *PH*(*R*) and $SH(N)=\{n_1,n_2,...,n_n\}$ as a linearly independent subset of *PH(M)* , the subset *SH*(*N*) is defined as a basis for *PH(M)* precisely when *SH*(*N*) is a *spanPH(M)* .

Example: If we have *PH(M)= PH(R³)* as a neutrosophic $R-$ module over a Ring *PH*(*R*), and if we take the sub set:

$SP(N)=\{n_1=(I,0,0),n_2=(0,I,0),...,n_n=(0,0,I)\}$ we can found it as a basis for *PH(M)* .

Free Neutrosophic module

If we have *PH*(*M*) as a neutrosophic $R-$ module over a ring *PH*(*R*) , *PH*(*M*) is said to be free neutrosophic module when it has a basis.

for examples:

• *PH*(*R*) itself is a free neutrosophic $R-$ module, having a basis element $1_{PH(R)}$.
• The zero neutrosophic $R-$ module 0 is a free neutrosophic $R-$ module with empty basis.
• $PH(M)=PH(\square^3)$ is a free neutrosophic $R-$ module with basis *SH*(*N*) in Example 3.5.5 .

Theorem: If we have *PH*(*M*) as a neutrosophic $R-$ module over a ring *PH*(*R*), the basis of *PH*(*M*) over *PH*(*R*) are like the basis of *M* over a *R*.

Proof: Let $N=\{n_1,n_2,...,n_n\}$ be any basis for *M* over *R* . We will proof *N* is a basis of *PH*(*M*):

1) We will show *N* is a linearly independent set in *PH*(*M*) : let $\alpha_1=k_1+m_1I$, $\alpha_2=k_2+m_2I$,…, $\alpha_n=k_n+m_nI \in PH(R)$,if $\alpha_1n_1+\alpha_2n_2+...+\alpha_nn_n=0 \Rightarrow k_1n_1+k_2n_2+...+k_nn_n=0$, $m_1n_1+m_2n_2+...+m_nn_n=0$

We have $k_i= 0$ and $m_j= 0$ where $i,j=0,1,2,...,n \Rightarrow \alpha_i=0, i=1,2,...,n$. This explains that *N* is a linearly independent set in *PH*(*M*).

2) We will show *N* that *spanPH(M)*:

let $n=a+bI =\alpha_1n_1+\alpha_2n_2+...+\alpha_nn_n$ .Then we have $a=k_1n_1+k_2n_2+...+k_nn_n$, $b=m_1n_1+m_2n_2+...+m_nn_n$

$a,b\in M \Rightarrow n=a+bI$ can be formed uniquely as a linear combination of $\{n_1,n_2,...,n_n\}$.

Theorem: If we have *PH*(*M*) as a (NM) over ring *PH*(*R*) , then the basis of strong (NM) is contained in the basis of the weak (NM).

Neutrosophic Module Homomorphism

If we have *PH*(*M*) and *PH*(*N*) as two neutrosophic $R-$ modules over a ring *PH*(*R*), a mapping $\varphi: PH(M)\to PH(N)$ (*PH*(*M*) into *PH*(*N*)) is said to be a neutrosophic $R-$ module homomorphism (NMH) , precisely when:

1) $\varphi(rm+r'm')=r\varphi(m)+r'\varphi(m)$ for all $m,m'\in M$ and $r,r'\in R$.

2) $\varphi(I)=I$.

Remarks:

• φ is said to be a neutrosophic $R-$ module monomorphism precisely when it is one-one.
• φ is said to be a neutrosophic $R-$ module epimorphism precisely when it is onto.
• φ is said to be neutrosophic $R-$ module isomorphism precisely when it is one-one and onto.
φ If is a neutrosophic $R-$ module isomorphism, then the invers $\varphi^{-1}: PH(N)\to PH(M)$ is also neutrosophic $R-$ module isomorphism and we write $PH(M)\cong PH(N)$.

Example: The mapping $PH(M)\cong PH(N)$ defined by $z(m)=0_{PH(N)}$ for all $m\in PH(M)$ is a neutrosophic $R-$ module homomorphism since $I\in M(I)$ but $z(I)\neq 0$, called the zero neutrosophic $R-$ module homomorphis.

The Kernel And The Image Of Neutrosophic $R-$ Module Homomorphism

If we have *PH(M)* and *PH(N)* as two neutrosophic $R-$ modules over a ring *PH(R)*, and if we take $\varphi : PH(M) \to PH(N)$ as a neutrosophic $R-$ module homomorphism, then:

- The kernel of φ referred to as kerφ is defined by the set $\ker\varphi = \{m \in PH(M) : \varphi(m) = 0\}$.
- The image of φ referred to as Im(φ) is defined by the set $\mathrm{Im}(\varphi) = \{n \in PH(N) : \varphi(m) = n \text{ for } m \in PH(M)\}$.

Example: If we have *PH*(*M*) and *PH*(*N*) as two neutrosophic $R-$ modules over a ring *PH*(*M*) and if we take $\varphi : PH(M) \to PH(N)$ defined by the form $\forall m \in PH(M) : \varphi(m) = m$, then:

1) φ is neutrosophic $R-$ module homomorphism.
2) $\ker\varphi = \{0\}$.
3) $\mathrm{Im}(\varphi) = PH(M)$

Theorem: If we have *PH(M)* and *PH(N)* as two neutrosophic $R-$ modules over a ring *PH(R)* and if we take $\varphi : PH(M) \to PH(N)$ as a $R-$ neutrosophic module homomorphism, then:

1) ker φ is not a neutrosophic submodule of *PH*(*M*) but a submodule of *M*.
2) Im φ is a neutrosophic submodule of *PH*(*M*).

Proof: 1) In order for ker φ to be a strong neutrosophic submodule of *PH*(*M*) it must contained *I* but $\varphi(I) \neq 0$ that means $I \notin \ker\varphi$. As we have in the classical case ker φ is a submodule of *M* .

Theorem: If we have *PH*(*M*) as a neutrosophic $R-$ module over a ring *PH*(*R*), the basis of *PH*(*M*) over *PH*(*R*) are like the basis of *M* over a *R* .

2) It is clear from the definition of Im φ .

Theorem: If we have *PH(M)* as a neutrosophic $R-$ module over a ring *PH(MR)* and if we take *SH(N)* as a submodule of *PH(M)*, then the mapping $\varphi : PH(M) \to PH(M)/SH(N)$ defined by $\varphi(m) = m + SH(N)$ for all $m \in PH(M)$ is not a neutrosophic $R-$ module homomorphism.

Proof : Through the conditions of the neutrosophic $R-$ module homomorphism, it must be $\varphi(I) = I$, but we have $\varphi(I) = I + SH(N) = SH(N) \neq I$. As a result, is not a neutrosophic $R-$ module homomorphism.

The Restriction of Neutrosophic Module Homomorphism

If we have *PH*(*M*) and *PH*(*N*) as two neutrosophic $R-$ modules over a ring *PH(R)* . If $\varphi : PH(M) \to PH(N)$ is a neutrosophic $R-$ module homomorphism and *SH*(*A*) be a submodule of *PH*(*M*), the neutrosophic $R-$ module homomorphism $\sigma : SH(A) \to PH(N)$ given by : $\forall a \in SH(A) : \sigma(a) = \varphi(a)$

σ Is defined as a restriction of φ over *SH*(*A*) .

We can notes:
1) σ is a neutrosophic $R-$ module homomorphism.
2) $\ker\sigma = \ker\varphi \cap SH(A)$.
3) $\mathrm{Im}\,\sigma = \varphi(SH(A))$.

Remark: If we have *PH*(*M*) and *PH*(*N*) as two neutrosophic $R-$ modules over a ring *PH*(*R*) and if we take $\sigma, \varphi : PH(M) \to PH(N)$ as two neutrosophic $R-$module homomorphisms, then : $(\varphi + \sigma)$ and $(\alpha\varphi)$ are not neutrosophic $R-$ module homomorphisms.

Proof : Through the conditions of the neutrosophic $R-$ module homomorphism, it must be $(\varphi + \sigma)(I) = I$ but we have $(\varphi + \sigma)(I) = \varphi(I) + \sigma(I) = I + I = 2I \neq I$ and must be $(\alpha\varphi)(I) = I$ but we have $(\alpha\varphi)(I) = \alpha\varphi(I) \neq I$ for any $\alpha \in PH(R)$. Therefore, $(\varphi + \sigma)$ and $(\alpha\varphi)$ are not neutrosophic $R-$ module homomorphisms.

Note: As a result of the remarks above, the set of all neutrosophic $R-$ module homomorphisms from *PH*(*M*) into *PH*(*N*) is not neutrosophic $R-$ module homomorphisms over *PH*(*R*) that means we have a different case from the classical $R-$ module

The Composition Of Tow Neutrosophic $R-$ Module Homomorphisms

If we have *PH*(*M*) , *PH*(*N*) and *PH*(*A*) as three neutrosophic $R-$ modules over a ring *PH*(*R*) and if $\varphi : PH(M) \to PH(N)$, $\sigma : PH(N) \to PH(A)$ as two (NMH), then the composition $\sigma \circ \varphi : PH(M) \to PH(A)$ is defined as:

$$\forall m \in PH(M) \;\Rightarrow\; (\sigma \circ \varphi)(m) = \sigma(\varphi(m)).$$

Note: $\sigma \circ \varphi : PH(M) \to PH(A)$ is also a neutrosophic $R-$ module homomorphism.

Proof: Through the conditions of the neutrosophic $R-$ module homomorphism, it must be $(\sigma \circ \varphi)(I) = I$. If we take $m = I \in PH(M)$, then:

$$(\sigma \circ \varphi)(I) = \sigma(\varphi(I))$$

$$= \sigma(I)$$

$$= I$$

As a result $\sigma \circ \varphi$ is a neutrosophic $R-$ module homomorphism.

Corollary: If we have σ, φ, μ as three neutrosophic $R-$ module homomorphisms, from *PH*(*M*) into *PH*(*M*) , then :

$$\sigma \circ (\varphi \circ \mu) = (\sigma \circ \varphi) \circ \mu$$

Theorem: If we have *PH*(*M*) , *PH*(*N*) and *PH*(*A*) as three neutrosophic $R-$ modules over a ring *PH*(*R*) and if $\varphi : PH(M) \to PH(N)$, $\sigma : PH(N) \to PH(A)$ as two neutrosophic module homomorphisms, then:

1) If $\sigma \circ \varphi$ is one-one , then φ is one-one .
2) If $\sigma \circ \varphi$ is onto, then σ nis onto.
3) If σ and φ are one-one, then $\sigma \circ \varphi$ is one-one.

The Exact Sequence Of Neutrosophic $R-$ modules

If we have *PH*(*M*) , *PH*(*N*) and *PH*(*A*) as three neutrosophic $R-$ modules over a ring *PH*(*R*) and if $\varphi : PH(M) \to PH(N)$, $\sigma : PH(N) \to PH(A)$ as two (NMH), then we say that the Sequence $PH(M) \xrightarrow{\varphi} PH(N) \xrightarrow{\sigma} PH(A)$ is an exact sequence, precisely when $\operatorname{Im}\varphi = \ker\sigma$.

Conclusıon

In this thesis we inspired from the neutrosophic philosophy which F.Smarandanche introduced the theory of neutrosophy in 1995. Basically we defined neutrosophic R-modules and neutrosophic submodules which are completely different from the classical module and submodule in the structural properties. It was shown that every weak neutrosophic R-module is a R-moduleand every strong neutrosophic R-module is a R-module. Finally, neutrosophic quotient modules and neutrosophic R-module homomorphism are explained and some definitions and theorems are given.

References

1. Zadeh LA (1965) Fuzzy sets. Inform Control 8: 338-40.`

2. Smarandache F (205) (t,i,f)-Neutrosophic Structures & I-Neutrosophic Structures. 8: 8.

3. Agboola AAA, Aklinleye SA (2014) Neutrosophic Vector spaces. Int J Math Comb 4: 20144.

4. Wang H, Smarandache F, Zhang Y, Sunderraman R (2014) Single Valued Neutrosophic Sets. 11-3.

5. Kandasamy WBV, Smarandache F (2006) Neutrosophic Rings, Hexis, Phoenix, Arizona.

6. Agboola AAA, Akwu AO, Oyebo YT (2012) Neutrosophic Groups and Neutrosophic Subgroups. Int J Math Comb 3: 1-9.

7. Agboola AAA, Akinola AD, Oyebola OY (2011) Neutrosophic Rings I. Int J Math Comb 4: 1-14.

8. Agboola AAA, Adeleke EO, Akinleye SA (2012) Neutrosophic Rings II. Int J Math Comb 2: 1-8.

9. Smarandache F (2003) A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability (3rd Edn). American Research Press, Rehoboth.

10. Kandasamy WBV, Smarandache F (2006) Some Neutrosophic Algebraic Structures and Neutrosophic NAlgebraic Structures. Hexis, Phoenix, Arizona.

11. Kandasamy WBV, Smarandache F (2004) Basic Neutrosophic Algebraic Structures and Their Applications to Fuzzy and Neutrosophic Models. Hexis, Church Rock.

12. Kandasamy WBV, Smarandache F (2006) Fuzzy Interval Matrices, Neutrosophic Interval Matrices and their Applications. Hexis, Phoenix, Arizona 1-9.