Journal of Biostatistics and Biometric Applications
Volume 3 | Issue 3
ISSN: 2455-765X

Research Article Open Access

Barycentric Coordinates for Polycons

Eugene L Wachspress*

East Windsor, New Jersy, USA

***Corresponding author:** Eugene L Wachspress, East Windsor, New Jersy, USA, E-mail: genewac@cs.com

Citation: Eugene L Wachspress (2018) Barycentric Coordinates for Polycons. J Biostat Biometric App 3(3): 308

Abstract

This discussion is limited to two-space although much of the analysis has been generalized to higher dimensions. Barycentric coordinates are needed in some applications for polygons with concave vertices. Such elements do not have rational barycentrics and mean-value coordinates (Floater, 2003) [1] are widely used. Any element with concave vertices may be replaced with a "polycon" which is an element with conics as well as lines as sides. There are rational barycentric coordinates for any "well-set" polycon where well-set is a generalization of convex. A concave vertex between two convex vertices may be replaced by a side node on a parabola through the three vertices. For example, star polygons have alternating convex and concave vertices. A star polygon may thus be replaced by a polycon with only parabolic sides. In general, when two or more concave vertices separate convex vertices the lines between the two convex vertices may be replaced with least-squares fit to a conic. Theoretical foundations and MATLAB implementation described here are in the GitHub site genepol/barycentric.

Keywords: Barycentric Coordinates, Polycons

Introduction

The advent of high speed digital computers in the 1950's revolutionized scientific studies. Formerly intractable solutions to complex mathematical equations could now be approximated to prescribed accuracy. One ubiquitous approach (Zienkiewizc, 1971)[2] was the "finite element method" (FEM). A variational analysis reduces each problem to solution for values at a finite number of points. This study is confined to two space dimensions. One partitions the planar region of concern into a patchwork of contiguous elements with a finite number of (x,y) nodes. A set of values equal to the number of variables for which a solution is desired is assigned to each node. A basis function is associated with each node. These bases over each element interpolate nodal values to yield values for the variables within the element. Initial FEM partitions were restricted to triangles and rectangles. Nodes were chosen to yield precise approximation within each element of polynomial solutions up to a prescribed degree. For example, linear functions require nodes only at element vertices. Degree-two approximation requires an additional node on each side. Many FEM problems required global continuity and this restricted behavior of bases on element boundaries. The basis functions (often called coordinates) for linear precision on a triangle are its barycentric coordinates. The basis functions for a rectangle are bilinear. These are the barycentric coordinates. The term barycentric may be associated with degree-one bases over an element. Polygon polynomial coordinates exist only for triangles and parallelograms. These were the only known barycentric coordinates until the introduction for general quadrilaterals of rational coordinates (Wachspress, 1971) [3]. This was generalized to rational coordinates for convex polygons with any number of sides (Wachspress, 1975) [4]. (Rational coordinates were also developed in that work for elements having curved sides) This work drew heavily on algebraic-geometry theory (Walker, 1962) [5]. This area of mathematics was virtually unknown to applied mathematicians at that time. There was little application for the next twenty years. Subsequent application to graphics problems and more sophisticated FEM problems led to more widespread application. The mean-value generalization to concave polygons (Floater, 2003) [1] is currently favored because of the removal of the convexity constraint.

An early FEM generalization to elements with curved sides was with isoparametric coordinates (Zienkowitz, 1971)[2], Degree-two triangle and square baycentrics in "local" coordinates were used to transform linear local sides into parabolic sides in "global" (x,y) coordinates. The barycentric coordinates were not generated explicitly. The FEM discrete equations were determined with integrals evaluated in local coordinates weighted by transformation jacobians. Values of the barycentrics in any given local coordinates could be transformed into those values at the corresponding (x,y) coordinates. This approach was widely used in major FEM programs and precluded application of the global rational coordinates introduced in 1975. Most modern analysis and application (other than isoparametrics) are restricted to polygons. The early analysis of rational coordinates was updated in (Wachspress, 2016)[6].

Review of barycentric coordinates for polygons

The symbol $(j_1; j_2;...; j_t)$ denotes a curve defined by distinct points j_s. For a line t = 2, for a conic t = 5, for a cubic t = 9, etc. The symbol $(j_1; j_2;...; j_t)$ with semi-colons instead of commas denotes the polynomial of lowest degree in x and y that vanishes on the curve. For any u(x,y) the symbol u_s is u normalized to unity at point s and the symbol $[u]_s$ is u evaluated at point s. Thus, $u_s = \frac{u}{[u]_s}$. (Figure 1) A polygon with n sides is an "n-gon". Polygon vertices are indexed in ccw order. All indices are mod n. This notation exposes relationships between curves and polynomials that vanish on these curves which are distinguished only by commas and semi-colons. Concepts in algebraic-geometry essential for generalization to elements with curved sides are exposed by this notation.

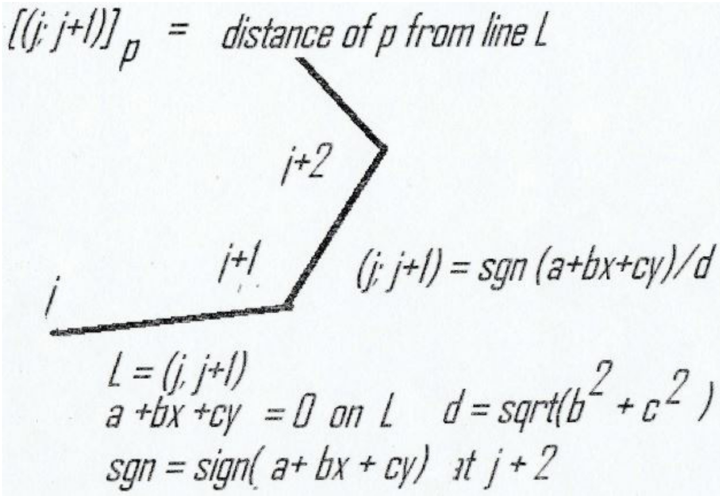

Figure 1: Poly Nomenclature

Barycentric coordinates for an element that is one of a contiguous non-overlapping set of elements must interpolate nodal values with linear precision within each element and attain global continuity. The barycentric coordinate associated with node j has the following properties:

1. $B_j(x; y)$ is bounded and continuous within the element.
2. $B_j(k) = \delta(j,k)$
3. $B_j(x; y)$ is zero on all "opposite" sides (the n - 2 sides other than the two adjacent sides);
4. $B_j(x; y)$ is linear on "adjacent" sides *(j-1, j)* and *(j, j+1).*
5. $[1\ x\ Y] = \sum_j [1\ x_j\ y_j] B_j(x, y)$.

Property 2 is a discrete orthonormality property. An element is not considered in isolation. It is often one element in a partition of a region into non-overlapping contiguous elements. Barycentric coordinates interpolate within the element nodal values j of a function f into an approximation $H(x,y) = \sum_j f(x_j, y_j) B_j(x,y)$ with $H = f$ when f is linear (degree-one interpolation of property 5.) Global continuity requires that vertex values off a side not affect the interpolation on the side (property 3) and the interpolant be the same on elements sharing a side (property 4). A linear function on *(j, j+1)* is uniquely defined by its values at vertices *j* and *j+1.* These properties are necessary and sufficient for the coordinates to provide a vertex basis for continuous, piecewise degree-one, approximation over a non-overlapping polygon partition of a bounded region ***R****(x,y).*

That barycentric coordinates for triangle [1, 2, 3] are

$$B_1 = (2;3)_1,\ B_2 = (1;3)_2,\ and\ B_3 = (1;2)_3.$$

has been known for many years. That these linear coordinates satisfies the five conditions is apparent.

Barycentrics for parallelogram [1, 2, 3, 4] are:

$$B_1 = (2;3)_1 (3;4)_1,\ B_2 = (3;4)_2 (4;1)_2,\ B_3 = (4;1)_3 (1;2)_3,\ and\ B_4 = (1;2)_4 (2;3)_4.$$

These coordinates are shown in (Figure 2).

Only for parallelograms where the opposite sides are parallel do these bilinear coordinates reduce to linear on adjacent sides. For example, (3;4) is constant on side (1,2) and (2;3) is constant on side (4,1). That property 5 is satisfied requires a simple observation. The bilinear approximation of f by H is precise on the boundary (with simple components) of order four. The only polynomial of degree two that vanishes on this curve is the zero polynomial. Hence, H = f within the n-gon. These coordinates have also been known for many years. Triangles and rectangles are ubiquitous in Finite-Element computation.

That polynomial barycentrics only apply to triangles and parallelograms was noted in early work. Suppose two sides of an element meet at non-vertex point p. Values at vertices are unrestricted. The linear variation on the sides will in general yield different values at p. Polynomials are single valued. Triangles and parallelograms are the only polygons that have no non-vertex intersections. Polynomial barycentrics cannot exist for any element with a non-vertex intersection of its sides.

In an attempt to broaden the class of elements for which barycentric coordinates exist, the obvious generalization was to rational bases. These were introduced in (Wachspress, 1971, 1975) [4][1].

The first new element was the general quadrilateral (which includes rectangles and trapezoids) best addressed in the projective plane where the line at infinity is displayed (Figure 2) as the "absolute line". The bilinear rectangle bases are divided by the linear form that vanishes on the line through the two exterior intersection points (eip). The denominator of any rational basis must contain these eip and this is the simplest. That the unique polynomial of minimal order containing all eip suffices will be established for all convex n-gons. For a parallelogram both eip are on the absolute line whose linear form in the affine plane is unity, for a trapezoid one of the eip is on the absolute line, and when the element has no parallel sides neither eip is on the absolute line. That these are indeed barycentric coordinates will be demonstrated in the following analysis of the general convex n-gon.

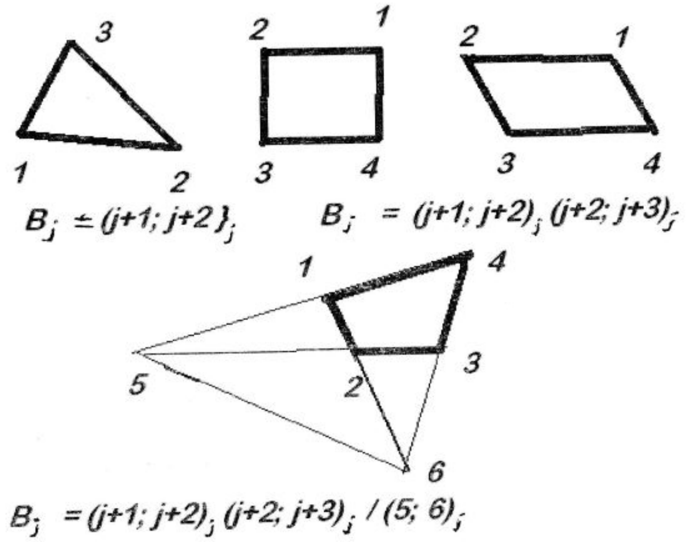

Figure 2: Triangles Parallelograms and Quandrilaterals

The n sides of an n-gon meet at n vertices and their extensions meet at n(n-1)/2 - n = n(n-3)/2 discrete double points (the eip) of the boundary curve. A curve of order n-3 has n(n-3)/2 degrees of freedom and these eip determine a unique curve Q. Polynomial vanishes at these eip. The denominators of the rational n-con barycentric coordinates must vanish at these points. Thus, the polynomial of least degree that can be a candidate for a common denominator.

In the seminal work (Wachspress, 1975) [4] this essential property was shown to be sufficient. For large n, finding the eip and generating Q from them required non-trivial computation. However, the existence of barycentric coordinates with this unique Q was established. More recent analysis eliminated the need to generate Q directly from the eip.

The construction of rational barycentrics for convex polygons is now reasonably simple. The barycentic coordinate at vertex *j* is $W_j = \frac{N_j}{Q}$ where Q is the same for all *j*. The linear forms of the sides are normalized to be positive within the n-gon. The n-gon boundary curve $\acute{\Gamma}$ of order n is the product of the linear forms of the n sides. Let $G_j = \frac{\acute{\Gamma}}{(j-1;j)(j;j+1)(j+1;j+2)}$. Let k_j be a normalization parameter to be determined. Then the product of the linear factors of the sides opposite j is $F_j = (j+1; j+2)G_j$ and $N_j = k_j F_j$ satisfies properties 1 and 3. Similarly, $F_{j+1} = (j-1; j)G_j$. The denominator is $Q = \sum_j N_j$. Then

$$W_j = \frac{k_j F_j}{Q}$$

satisfies property 2 and Q has been chosen so that the sum of the W_j is unity as required by property 5. The k_j are chosen to satisfy property 4, linearity on adjacent sides. On side *(j, j+1)*, the only nonzero coordinates are W_j and W_{j+1}. Thus, dividing numerator and denominator by the common factor G_j:

$$W_j = \frac{k_j F_j}{k_j F_j + k_{j+1} F_{j+1}} \equiv \frac{k_j(j+1; j+2)}{k_j(j+1; j+2) + k_{j+1}(j-1; j)}$$

The numerator is linear. Setting $k_1 = 1$, the remaining k_j may be found recursively so that the denominator is constant:

[1]Graphic barycentric coordinates appear as wedges and I denoted them by W in my early work. Now they are called "Wachspress coordinates" so the W was fortuitous, although I was not unaware of this possibility at the outset. The theory has now been described in many papers and will not be repeated here. My recent book (Wachspress, 2016) [6] reproduces the 1975 book as Part 1 and more recent work as Part 2.

$k_j\left[(j+1;j+2)\right]_j = k_{j+1}\left[(j-1;j)\right]_{j+1}$ and $k_{j+1} = k_j \dfrac{\left[(j+1;j+2)\right]_j}{\left[(j-1;j)\right]_{j+1}}$

The linear basis function $B_{j+1} = (j-1;j)_{j+1}$ and $B_j = (j+1;j+2)_j$ sum to unity on S_j.

The numerator at j is $k_j(j+1;j+2)$ and the denominator is $k_j[(j+1;j+2)]_j$. The ratio is $W_j = B_j$.

The numerator at $j+1$ is $k_{j+1}(j-1;j)$ and the denominator is $k_{j+1}[(j-1;j)]_{j+1}$. The ratio is $W_{j+1} = B_{j+1}$. Linearity on the sides is established[2]. The algorithm may be continued to compute k_{n+1} from k_n. This value should be equal to $k_1=1$ and the difference $k_{n+1}-1$ should be due only to rounding error. That this is true is assured by the established uniqueness of barycentric coordinates for any convex n-gon.

However, a direct proof is comforting. Closure is independent of the normalization of the linear forms $a+bx+cy$ that vanish on the sides. If we set $b^2+c^2 = 1$, then $[(j;j+1)]_v$ is the distance of vertex v from side $(j, j+1)$. Let $h_1(j) = [(j+1;j+2)]_j$ and $h_2(j) = [(j-1;j)]_{j+1}$. Then $\frac{k_{j+1}}{k_j}=\frac{h_1(\mathrm{j})}{h_2(\mathrm{j})}$ Let A_j be the n-gon exterior angle (acute since the n-gon is convex) at vertex j. Then (Figure 3) $\frac{k_{j+1}}{k_j}=\frac{\sin A_j}{\sin A_{j+1}}$ and $\frac{k_{j+1}}{k_1}=\frac{\sin A_1}{\sin A_{j+1}}$ When $j = n$ closure is established since j is mod n.

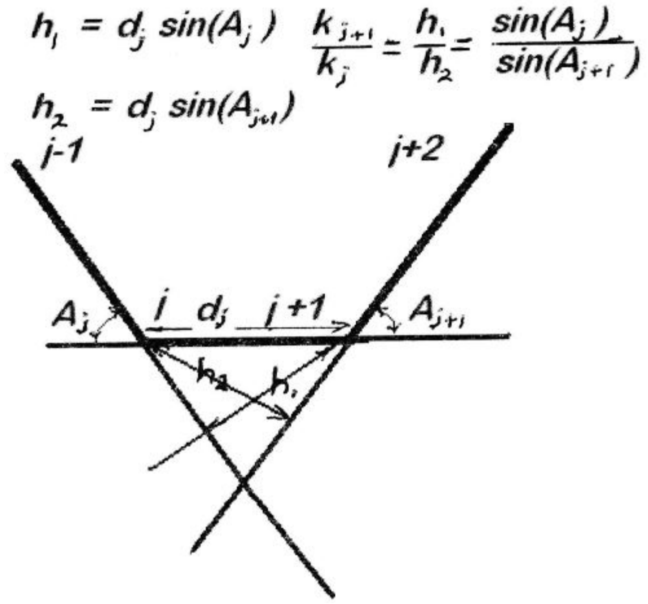

Figure 3: GADJ

All the factors in this analysis are positive within the n-gon so the coordinates are positive inside the boundary. This positivity is sometimes added to the properties of barycentric coordinates. It is not satisfied in the generalization to elements with curved sides. The only remaining property to be verified is 5. When f is linear, H - f = 0 on the boundary which is an irreducible curve of order n. Within the n-gon,

$$H - f = \frac{\sum_j f_j N_j(x,y) - f(x,y)Q(x,y)}{Q(x,y)}$$

The numerator is a polynomial of maximal degree n-2 and the denominator is positive and bounded within the n-gon. The numerator vanishes on the boundary which is a curve of order n. Therefore, the numerator must be the zero polynomial. All five properties have now been verified and these W_j are barycentric coordinates for the n-gon.

The denominator Q is the sum of numerators of degree n-2. Theory establishes that the curve **Q** on which Q = 0 is the curve of minimal degree that contains all the external intersection points (eip) of the boundary components. This curve is of maximal order n-3. The order is less when parallel sides (which meet at infinity) are such that the "line at infinity" (absolute line in projective coordinates) is in curve **Q**. This curve is of order n-3 in the projective plane. The linear form of the absolute line becomes unity in the affine plane and the result is that Q is of degree less than n-3. An example is the regular hexagon which has three pairs of parallel sides. In this case Q is a circle of order n-4 = 2 instead of n-3. Alternative constructions of barycentric coordinates for convex polygons do not identify these coordinates as the ratio of polynomials of degree n-2 and a common denominator of degree not greater than n-3, nor do they ever consider the fact that the numerators vanish at all the eip. In fact, there is no mention of eip. This renders generalization of these constructions to elements with curved sides extremely difficult.

[2]This recursive determination of the k_j was introduced in (Dasgupta, 2003) [7] and was subsequently generalized to the GADJ algorithm for elements with curved sides in (Dasgupta and Wachspress, 2008) [8].

Polycons

A polycon may have linear and conic sides with at least one conic side. A polycon with n sides is an n-con. Barycentric coordinates for polycons were introduced 45 years ago (Wachspress, 1975) [4]. Dasgupta's recursion (known as GADJ) was generalized to elements with curved sides in 2008 (Dasgupta and Wachspress, 2008) [8]. Two new constructions are developed here. First is an altenative implementation of GADJ. Second is replacing of concave vertices with side nodes of parabolas. This replaces concave n-gons with n-cons which have rational barycentric coordinates. The order m of an n-con is the number of linear sides plus twice the number of conic sides. A conic side common to two elements is convex in one element and concave in the other. Restriction to convexity limits conic sides to convex region boundaries. Rational barycentrics exist for "well-set" elements. A well-set element need not be convex. It is well-set when all vertices are simple transverse intersections of adjacent sides and when no side contains any point off the side itself that is on the element boundary or interior to the element. A convex polygon is well-set. (Figure 4)

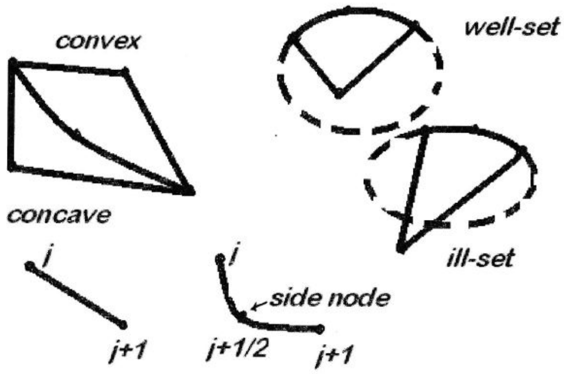

Figure 4: Polycon Concepts

A linear function has two degrees of freedom on a line. Vertex values suffice to determine linear behavior on any linear side. A linear function has three degrees of freedom on a conic side. A side node *j+1/2* is chosen on each conic side *(j, j+1/2, j+1)*. These three nodes do not determine a unique conic. The symbol just indicates that the side is a conic already defined. Similarly *(j; j+1/2; j+1)* is the quadratic that vanishes on the side. Two other points must be specified to define this quadratic. Barycentric coordinates for conics satisfy the five properties enumerated for convex polygons.

Property 3 requires each vertex numerator N_j to vanish on the sides opposite vertex *j*. When the adjacent sides are both linear F_j is of degree m-2, when one side is conic, F_j is of degree m-3, and when both sides are conic F_j is of degree m-4. Property 2 requires that N_j vanish at the adjacent side nodes. An "adjacent" factor P_j is introduced (Figure 5). This factor must also vanish at the exterior intersections e of the adjacent sides. P_j at a line-line vertex is unity, at a line-conic vertex *(e; j-1/2) or (e; j+1/2)*, and at a conic-conic vertex $(e_1;e_2;e_3;\ j\text{-}1/2;j\text{+}1/2)$. Adjacent factor P_j is normalized to unity at *j*. In all cases F_jP_j is of degree m-2. The adjacent factor at all side nodes is unity and the opposite factor is of degree m-2. Hence, all numerators are of degree m-2. The denominator Q is called the polycon "adjoint". It has properties in relationship to the polycon boundary curve which are similar to those of an algebraic-geometry adjoint polynomial. The sum of the degree m-2 numerators is of degree not greater than m-3 when the k_j and $k_{j+1/2}$ are determined with the linearity property. Once linearity on the sides is established, all five properties are satisfied.

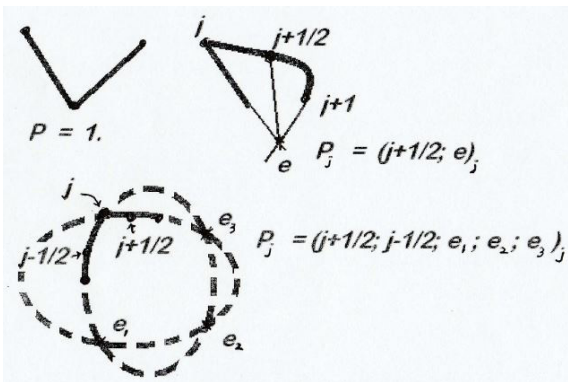

Figure 5: Adjacent factors

When the eip are distinct and there are none at infinity, computation of the adjacent factors is simple.

Otherwise one must allow for a variety of possibilities. The 600 line MATLAB program poly2018 computes barycentric coordinates for polycons. The first 300 lines contain a set of test problems and generate vertex, side node and eip coordinates. The next 100 lines determine the adjacent factors. The next 100 lines determine the numerator normalization parameters k_j and $k_{j+1/2}$ with the GADJ algorithm. The final 100 lines remove spurious terms in Q due to round-off error and specify the barycentric coordinates.

The major complication in passing from polygons to polycons is determination of the adjacent factors and the numerator normalization parameters.

Algebraic-geometry foundations

Algebraic-geometry foundations essential for rational barycentric construction were developed in (Wachspress, 1975, 2016) [4,6]. The set of points where algebraic curves of orders p and q intersect are in the "divisor" of these curves. When all intersections are simple these are the *pq* elements of the divisor. When a point is at a common tangent of the curves or at a multiple point on either curve there are fewer than pq points. Divisor theory introduces elements which expand the intersection points into a set of *pq* divisor elements. Conics do not have multiple points. The full theory of divisors is not needed for application to polycons. Let P be the polynomial of least degree that vanishes on algebraic curve **P**. The symbol **P.Q** denotes the divisor of curves **P** and **Q**. In this application, **P.Q** is just the set of intersections of curves **P** and **Q** and consists of vertices and eip (external intersection points). The assertion that **P** ≡ **Q** mod **S** (that is, "P is congruent to R mod S ") means that there is a scalar b such that P - bR = 0 on curve **S**. The crucial algebraic-geometry theorem is:

DIVISOR THEOREM: If **P . S = R . S,** then P ≡ R mod **S**.

This deceptively simple theorem provides a powerful basis for construction of rational barycentrics. It enables analysis replacing barycentric coordinate factors along a side with equivalent factors so that factors common to numerator and denominator may be cancelled to establish linear rather than rational variation on a side. This plays a crucial role in the GADJ algorithm for computing the numerator normalization parameters k_j and $k_{j+1/2}$.

GADJ for polycons

The seminal construction of barycentric coordinates in (Wachspress, 1975) [4] included computation of Q_{m-3} directly from the m(m-3)/2 multiple points of the extended element boundary curve **Γ́**. There are *m(m-3)* elements in **Q.T**. These are precisely the eip from which **Q** is generated. Thus, Q cannot vanish on the boundary of the element and may be normalized to be positive there. It has been conjectured that Q does not vanish within any well-set element. Although this has yet to be proved, no counter-example has been found. Satisfaction of Property 1 rests on this conjecture[3].

The construction of Q was simplified with GADJ. Only the eip of adjacent sides are needed for computation of the adjacent factors. This obscures the fact that all the eip are included in **Q**. For each coordinate, the numerator kN = kFP where P = 0 at all the eip of the adjacent sides and F = 0 at all eip on the opposite sides. Since Q is the sum of these numerators, Q = 0 at all the eip.

The GADJ generalization to polycons described in (Dasgupta and Wachspress, 2008) [8] has been improved.

The polycon analysis proceeds as for polygons. Let S_r denote the linear or quadratic form that vanishes on side S_r. Then polynomial $G_j = \frac{\Gamma}{S_{j-1}S_jS_{j+1}}$ is common to the coordinates that do not vanish on side $\mathbf{S}_j$. When G_j is divided out of numerator and denominator, the variation on side S_j is:

$$W_j = \frac{k_jP_jS_{j+1}}{k_jP_jS_{j+1} + k_{j+1/2}S_{j-1}S_{j+1} + k_{j+1}P_{j+1}S_{j-1}}$$

where the second term in the denominator appears only for a conic side.

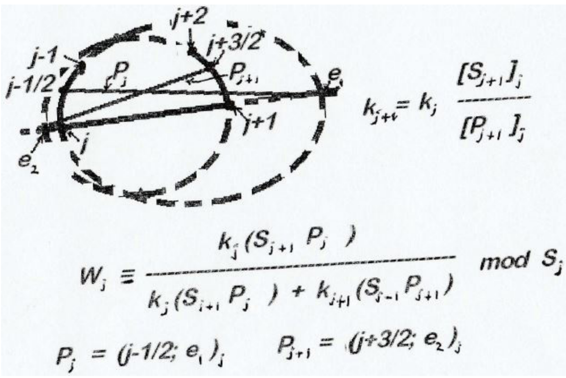

Figure 6: Polycon GADJ:linear(j, j+1)

Each term contains all the eip of $\mathbf{S}_j$ with $\mathbf{S}_{j-1}$ and $\mathbf{S}_{j+1}$. A unique curve **R** of least degree contains only these eip on side $\mathbf{S}_j$. This curve may be determined from the eip. This is complicated when some of the eip fall on the absolute line due to parallel lines or asymptotes. This complication has already been addressed in determining the adjacent factors. When $\mathbf{S}_j$ is linear (Figure 6) there

[3]In seeking a well-set counter-example of lowest order I have been able to prove that it must be convex with no linear sides. Although the algorithm has succeeded for all candidates thus far considered, a general proof is forthcoming

is no side node, $R = P_j P_{j+1} \; and \; \frac{P_j S_{j+1}}{P_j P_{j+1}} \equiv \frac{S_{j+1}}{P_{j+1}}$. The linear basis functions $B_{j+1} = (j-1; j)_{j+1}$ and $B_j = (j+1; j+2)_j$ sum to unity on $\mathbf{S}_j$. There is a subtle point preempted by this choice of R. When side $\mathbf{S}_{j+1}$ is a conic it intersects $\mathbf{S}_j$ at eip e_2 and vertex *j+1*. Therefore, should be congruent to *(j-1/2; e$_1$)(j+1; e$_2$)* or *(j-1/2; j+1)(e$_1$; e$_2$)*. Why then is R = P_j P_{j+1} The error is that *(e$_2$; j+1) = (j; j+1)* and vanishes on side $\mathbf{S}_j$. $\mathbf{S}_{j+1}$ is not congruent to *(e$_2$, j+1)* on side $\mathbf{S}_j$ This leads to the factor (e_1; e_2) = 0 on $\mathbf{S}_j$ with a result of R = 0. P_{j+1} captures e_2 as its only intersection with $\mathbf{S}_j$.

To apply the Divisor Theorem and find a recursion formula for k, we note that

$$\mathbf{P_j S_{j+1}} \cdot \mathbf{S_j} \text{ and } (\mathbf{j+1; j+2})\mathbf{R} \cdot \mathbf{S_j} = e_j, e_{j+1}, j+1.$$

Therefore, on side $\mathbf{S}_j, P_j S_{j+1} = c_j (j+1; j+2) R = c_j [(j+1; j+2)]_j B_j R$ and we define $b_j \equiv c_j [(j+1; j+2)]_j$. Since $[B_j]_j = 1 \; and \; [P_j]_j = 1$, the Divisor Theorem yields $b_J = [S_{j+1} / P_{j+1}]_j$. Thus, $P_j S_{j+1} = b_j B_j (x, y)$ on side S_j.

Similarly, $\frac{S_{j-1}}{P_{j+1}} = b_{j+1} B_{j+1}$, where $b_{j+1} = [S_{j-1} / P_j]_{j+1}$. Since $B_j + B_{j+1} = 1$ and $[B_r]_s = \delta(r, s)$, the denominator $k_j b_j B_j + k_{j+1} b_{j+1} B_{j+1} = k_j b_j$ when $k_{j+1} = k_j \frac{b_j}{b_{j+1}}$.

The situation is more complicated when $\mathbf{S}_j$ is conic (Figure 7). Note that ini Fig. 7 eip e_2 and e_3 are complex conjugates where $\mathbf{S}_{j-1}$ and $\mathbf{S}_j$ intersect. Intersections could also fall on the absolute line (at infinity). For example, two circles intersect at the ``polar'' points *(w,x,y) = (0,1,i)* and *(0,1,-i)*. The product $P_j P_{j+1}$ now has an added double point at *j+1/2*. It is actually one degree higher than R. The only linear form that the product can be divided by to remove this double point is the tangent to $S_j \; at \; j+1/2$. Let **T** be this tangent. Then $R = \frac{P_j P_{j+1}}{T}$. The barycentric coordinates for triangle $[j, j+1/2, j+1]$ are

$$B_j = (j+1/2; j+1)_j \quad B_{j+1/2} = (j; j+1)_{j+1/2} \quad B_{j+1} = (j; j+1/2)_{j+1}.$$

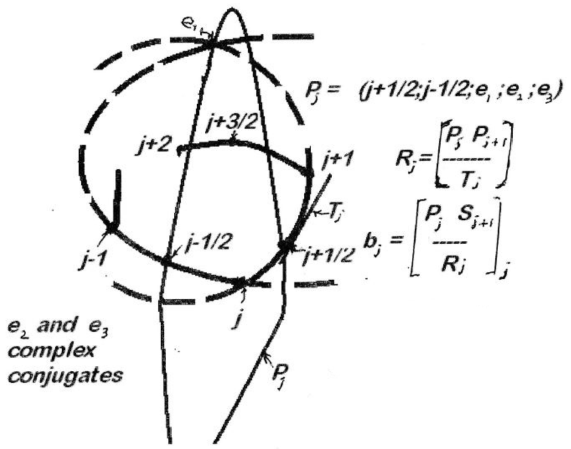

Figure 7: Polycon GADJ for a Conic side

These coordinates sum to unity. The divisors for the conic side are now

$$\mathbf{P_j S_{j+1}} \cdot \mathbf{S_j} \text{ and } (\mathbf{j+1/2; j+1})\mathbf{R} \cdot \mathbf{S_j} = e_j, e_{j+1}, j+1/2, j+1$$

$$\mathbf{P_{j+1} S_{j-1}} \cdot \mathbf{S_j} \text{ and } (\mathbf{j; j+1/2})\mathbf{R} \cdot \mathbf{S_j} = e_j, e_{j+1}, j, j+1/2$$

$$\mathbf{S_{j+1} S_{j-1}} \cdot \mathbf{S_j} \text{ and } (\mathbf{j; j+1})\mathbf{R} \cdot \mathbf{S_j} = e_j, e_{j+1}, j, j+1.$$

The Divisor Theorem then yields $P_j S_{j+1} = c_j (j+1/2; j+1) R$ and $b_j \equiv c_j [(j+1/2; j+1)]_j$. When $b_j = \left[\frac{S_{j+1}}{R}\right]_j$, $\frac{P_j S_{j+1}}{R} = b_j B_j$.

Similarly, when $b_{j+1} = \left[\frac{S_{j-1}}{R}\right]_{j+1}$,

$$\frac{P_{j+1} S_{j-1}}{R} = b_{j+1} B_{j+1}$$

For the side node, $S_{j-1} S_{j+1} = c\{j+1/2\}(j; j+1) R$ and $b_{j+1/2} \equiv c_{j+1/2} [(j; j+1)]_{j+1/2}$. When

$$b_{j+1/2} = \left[\frac{S_{j-1}S_{j+1}}{R}\right]_{j+1/2}$$

Dividing numerator and denominator in W_j by R, we find that the denominator is constant when $k_j\ b_j = k_{j+1/2}\ b_{j+1/2} = k_{j+1}\ b_{j+1}$ which gives the recursion:

$$k_{j+1} = k_j \frac{b_j}{b_{j+1}} \ and \ k_{j+1/2} = k_j \frac{b_j}{b_{j+1/2}}$$

Evaluating R at the vertices is straightforward. However, at the side node *j+1/2* the numerator P_jP_j+1 and the denominator T each have a double point. The value of R at *j+1/2* is computed with L'Hopital's rule. First $S_j = 0$ yields $y = f(x)$ or $x = f(y)$ on this side. The choice is made according to the slope of T. Let the derivative of any function V with respect to the retained variable be denoted by V'. Both adjacent factors vanish at *j+1/2*.

Thus, the second derivative of P_jP_{j+1} is $U \equiv 2P_j'P_{j+1}'$. The second derivative of T which is no longer linear with respect to the retained variable is T'. $[R]_{j+1/2} = UT_{j+1/2}$. On side $\mathbf{S}_j$, $W_j = \frac{k_jb_jB_j}{k_jb_j} = B_j$.

Similarly, $W_{j+1/2} = B_{j+1/2}$ *and* $W_{j+1} = B_{j+1}$.

Parabola replacement of concave vertices

Star polygons often occur in graphics application. These polygons have alternating convex and concave vertices. Such elements do not have rational barycentric bases. Mean-value coordinates are well suited for star polygons. An alternative approach enabling rational barycentric coordinates is to replace concave Vs with parabolas with the concave polygon vertices as parabola side nodes. This was suggested in (Wachspress, 2016) [6]. These star polycons have only parabolic sides. Application of theory for construction of barycentric coordinates to elements with curved boundaries has been sparse. Star polycons provide impetus for new consideration.

In general there are polygons and polycons with concave vertices that may be replaced with side nodes of parabolas. Contiguous elements must be constructed with the same curved sides (even though convex vertices may be replaced by side nodes.) In comparing rational and mean-value barycentrics a few properties warrant consideration. For convex polygons there are no significant differences. As an interior angle approaches 180° both approaches lead to large gradients at the vertex. When the offending vertex is replaced with a parabola side node the gradient is well behaved. In the limit the parabola reduces to a line with a side midpoint and degree-two approximation on the side. As the angle passes through 180° the element becomes concave and rational barycentrics exist only for the parabolic replacement. The rationals have smaller gradients when the angle is near 180°. Mean-value barycentrics have not been developed for elements with curved sides. Mean-value coordinates can also treat adjacent concave vertices. Replacement of linear sides connecting these vertices with one or more conic sides yields elements amenable to rational barycentric coordinates.

Rational polycon barycentrics are not positive. However, positive approximation can be ensured by increasing values at side nodes to maintain positivity on the boundary and introducing an interior hat basis function to maintain positivity within the element. There are problems where large gradients are a physical property and should be retained as, for example, in stress computations where cracks form at sharp corners. Corners may be rounded to reduce crack formation. The mean-value coordinates are preferable for an element with sharp corners while rational bases with curved sides are preferable for a rounded element. The purpose of this note is not to suggest side nodes of parabolas as a ubiquitous replacement for concave vertices but rather to introduce this option.

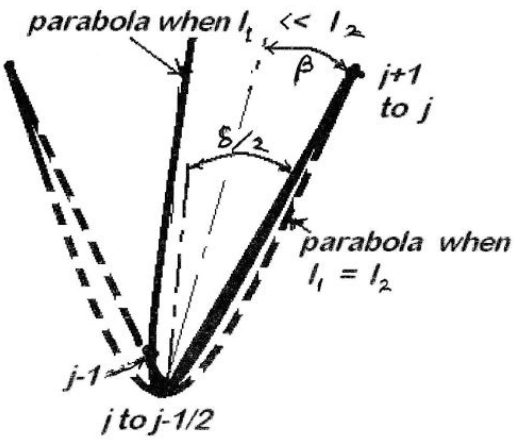

Figure 8: Polygon V to Polycon Parabola

The first new challenge is construction of a parabolic replacement of a concave V (Figure 8). The isoparametric parabola suggested in (Wachspress, 2016) [6] is not best. The concave vertex is in general not at the vertex of the isoparametric parabola, but there is a unique parabola whose vertex is the replaced concave vertex. As the ratio of the lengths L1 and L2 of the arms of the V increase, the axis angle with the long side decreases. Let the V vertices be 1, 2 and 3 with 2 concave. If coordinates *(xp,yp)* are chosen with yp along the parabola axis and (0,0) as the parabola vertex, the parabola $yp = c \cdot xp^2$ requires $[\frac{xp^2}{yp}]_1 = [\frac{xp^2}{yp}]_3$. If δ is the interior angle of the V and β is the angle between the y-axis and the longer side, L_2, then for point 2 to be at the apex of a parabola through the three vertices:

$$V_\beta \equiv L_1[\frac{1}{\cos(\delta-\beta)} - \cos(\delta-\beta)] - L_2[\frac{1}{\cos(\beta)} - \cos(\beta)] = 0$$

One may solve this transcendental equation analytically or approximate the solution by interpolation on a table of $V_\beta \, for \, \beta \in [0, \delta/2]$. This parabola must then be rotated and translated to its actual *(x,y)* position. Suppose $\delta = 90^o + \alpha$ for some positive $\alpha < 90^o$. For equal lengths the symmetry axis is at $\beta(r=1) = 45^o + \alpha/2$. As the ratio r of L_2/L_1 increases, $\beta(r)$ goes from $\beta(1)$ to a minimum of **α.** For example, when $\delta = 120^o$, the smallest $\beta = 30^o$.

A value of *r = 5* is satisfied with *β* around 45°. Here, *β* goes from 60° to 30° as r varies from unity to infinity. When adjacent parabolic sides have parallel axes, two of the eip are on the absolute line (that is, at infinity). If the parabolas intersect at a point closer to the side node than the vertex, the vertex is moved to this point and the former vertex becomes an eip. (Figure 9b) In rare cases the parabolas may be tangent at the vertex (Figure 9c). This leads to an interesting study establishing stability of its barycentric coordinates. This element is said to be "essentially well-set".

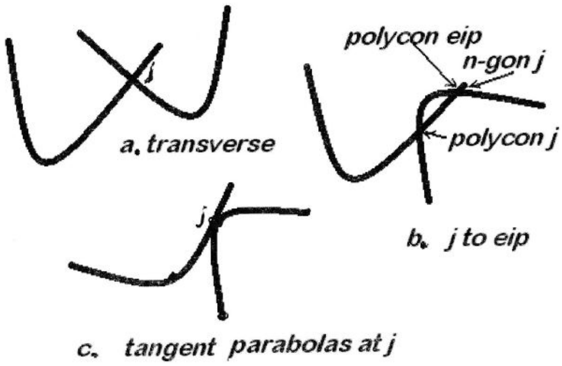

Figure 9: Intersecting Parabolas at a Vertex

Essentially well-set elements

In three space dimensions, convex polyhedra restricted to vertices of order three were introduced in (Wachspress, 1975) [4]. Although the adjoint vanishes to order p - 3 where p is the order of the vertex, (Warren, 1996) [9] realized that the singularity at the vertex is removable and constructed barycentric coordinates for all convex polyhedra. An alternative to Warren's construction was described in (Wachspress, 2010) [10]. Application to 2D elements was not considered. However, polycons with tangential intersection at parabola vertices may be considered. Such elements are designated here as "essentially well-set". That rational bases may be constructed for such elements is easily demonstrated by the approach followed in the 3D development. The adjacent factor at tangential vertex j vanishes at j and thus cannot be normalized to unity there. Normalization to unity at vertex j+1 suffices. The construction for a simple element may be instructive. (Figure 10) The vertices are:

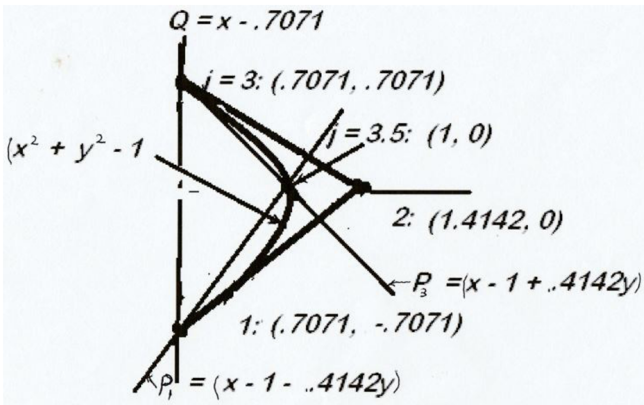

Figure 10: An essentially well-set element

$1=(1/\sqrt{2},-1/\sqrt{2});2=(\sqrt{2},0);3=(1/\sqrt{2},1/\sqrt{2})$. Sides S and adjacent factors P for the 3-con of order four are: $S_1=\sqrt{2}+y-x, S_2=\sqrt{2}-x-y, S_3=x^2+y^2-1, P_1=x-(\sqrt{2}-1)y-1$, and $P_3=(\sqrt{2}-1)y+x-1$. The adjoint is $Q=\sqrt{2}x-1$. The rational basis functions other than at 1 and 3 are standard. The basis at 3 is

$$W_3=\frac{(\sqrt{2}+y-x)[(\sqrt{2}-1)y+x-1]]}{(2-\sqrt{2})(\sqrt{2}x-1)}.$$

On side S_2 we substitute $\sqrt{2}-x$ for y and reduce W_3 *to* $2-\sqrt{2}x$ on the side. Linearity on circular arc S_3 is easily demonstrated by showing the equivalence of the numerator and Q P_1.

If the extension of a line adjacent to a parabolic side hits the parabolic side rather than its extension, the element is not well-set. This may be remedied by introducing a concave vertex at the midpoint of the linear side and perturbing the convex vertex at which the two sides intersect in order to replace this parabola-line vertex with a parabola-parabola vertex. Similarly, a parabola-parabola vertex may have to be perturbed to assure a well-set element. Programs poly2018 and starcon assume input leads to well-set elements without perturbation.

Adjacent concave vertices

When two or more concave vertices separate a pair of convex vertices a least squares conic fit to them may be constructed (Figure 11). Points on the midpoints of the sides between the convex vertices are added to the fitted points in the MATLAB conpq program. This yields seven points when there are only two concave points. Let t be the number of points to be fit to a conic.

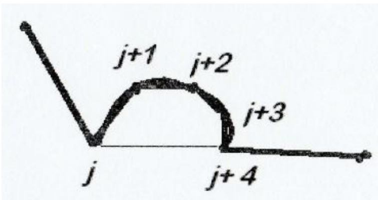

Figure 11: Least Square Conic fit to Successive Concave Vertices

In conpq the points are weighted. The convex vertices are weighted with $q_1 = q_t = 10$, the concave vertices with $q_p = 2$, and the midpoints with $q_p = 1$. This leads to relatively small movement of the convex vertices and reduces the influence of the side midpoints. This choice was arbitrary and a user is welcome to modify it. The curve is normalized to -1 at the midpoint of the line connecting the two convex vertices. The equation to be solved with a generalized inverse is $M\mathbf{z}=\mathbf{r}$ with M of order (t+1,6) and **r** a vector of length t+1. The first t elements in **r** are zero and the last element is -1. Row p of M is $q_p[1\ \ x_p\ \ y_p\ \ x_p^2\ \ x_py_p\ \ y_p^2]$. Let $G\equiv(M^\top M)^{-1}$. Then $\mathbf{z}=GM^\top\mathbf{r}$. The conic is $[1\ \ x\ \ y\ \ x^2\ \ xy\ \ y^2]\mathbf{z}$. A hyperbola has two branches. To avoid a second branch cutting through the element a hyperbola is replaced by a parabola. The vertex of the parabola is chosen as the point on the hyperbolic curve where the tangent is parallel to the line connecting the convex vertices. A conic is hyperbolic when its discriminant D=Z_5-4Z_4Z_6 is positive.

Adjoint positivity with higher order sides

An element with sides of higher order than two is called a "polypol". A curve of order *j* can have at most $(j-1)(j-2)/2$ double points when all multiple points are "blown up" by quadratic transformations.

The genus is the deficiency in double points. The genus of all conics is 0, Any curve which has a rational parametrization is of genus zero. An element whose sides are all of genus zero is said to be a "rational algebraic element."

A rational algebraic element of order m has a unique adjoint of order m-3 (Wachspress, 1975, 2016) [4,6]. The genus *g* of an element is the sum of the geni of the sides. It was observed in these references that g arbitrary exterior points could be used to define adjoints. It was conjectured that the adjoint of any well-set rational algebraic element did not vanish within the element. That this conjecture may not hold for the adjoint of an element of genus g greater than one will now be established. The adjoint of order m-3 intersects the element boundary of order *m* in *m(m-3)* points. Each of the *m(m-3)/2* points which define the adjoint is a double point of the boundary curve. Thus, the adjoint cannot intersect the boundary elsewhere. On the other hand, if g is not zero the arbitrary added points are no longer double points. When *g = 1* the adjoint cannot enter the element since it cannot have another exit point. The conjecture could in that case still is valid.

When *g* is greater than one, there could be boundary crossings, depending on the choice of points to determine the adjoint. Once the g added points are chosen, the GADJ algorithm may be applied to determine the corresponding adjoint curve.

If this curve does not vanish within the element, rational barycentric coordinates exist for the element. Lack of a well-defined unique adjoint suggests that elements that are not rational be avoided. Rational bases for elements with curved sides have not yet been applied. A possible application for conic sides has been presented in this note. Programs have yet to be written for more general polypols. If such elements are introduced, it would be advisable to restrict initial application to rational algebraic elements.

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