

# Symmetric Balanced Incomplete Block Designs for Seven Treatments from Mutually Orthogonal Latin Square of Order 7

Saka AJ<sup>1</sup>, Adeleke MO<sup>2</sup>, Adeleke BL<sup>2</sup> and Akeyede I<sup>3</sup>

<sup>1</sup>Department of Mathematics, Obafemi Awolowo University, Nigeria

<sup>2</sup>Department of Statistics, University of Ilorin, Nigeria

<sup>3</sup>Department of Mathematics, Federal University, Lafia, Nigeria

\*Corresponding author: Saka AJ, Department of Mathematics, Obafemi Awolowo University, Ile-Ife 220005, Nigeria, Tel: +2348033380233, E-mail: ajsaka@oauife.edu.ng; sakajamiu@gmail.com

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## Abstract

A set of Mutually Orthogonal Latin Squares (MOLS) of order 7 gives rise to series of incomplete block designs, such as; balanced incomplete block designs (BIBD), and partially balanced incomplete block designs of two, three, four, five and six associate classes, that is PBIBD( $k$ ) with  $k=2,3,4,5$  and 6. Two distinct Near-Resolvable BIBDs that are symmetric are equally obtained. All the aforementioned designs are constructed via orderly combinations of off-diagonal elements of a complete set of mutually orthogonal Latin squares (MOLS).

**Keywords:** Orthogonal Latin Square, Symmetric, Near-Resolvable, Incomplete Block Designs, Partially Balanced Incomplete Block Designs

## Introduction

Generally, Latin squares occur in many structures, including group multiplication tables, and calyey tables. To be precise Latin squares are simply referred to as the multiplication tables of an algebraic structure called a quasi-group. Latin squares exhibit interesting combinatorial structures that have a wide range of areas of applications, such as design of experiment, combinatorics, tournament scheduling, finite geometries, and cryptography. One of the commonly known examples of Latin squares is Sudoku squares. Incidentally, Latin squares were first studied by a Swiss mathematician Leonhard Euler in 1782, while the problem of arrangement of 36 military officers in a clearly unique square was addressed [1]. This question concerns a group of thirty-six officers of six different ranks, taken from six different regiments, and arranged in a square in a way such that in each row and column there are six officers, each of a different rank and regiment [2].

Eulers conjecture remained unsolved for over a hundred years before a step was taken towards confirming his predictions. In 1900 the French mathematician proved that there is no solution to the 36 officers problem by an exhaustive elimination of all possible cases, there by verifying Eulers conjecture for  $n = 6$  [3]. This still left the orders  $n = 10,14,18,\dots$  unresolved, and it would only be 59 years later, when Indian mathematicians Raj Chandra constructed a Graeco-Latin square of order 22, that a special case of Eulers conjecture was disproven for the first time [4]. By this time the term Graeco-Latin square was no longer in use; the concept of orthogonality between Latin squares had taken its place. Also in 1959, the American mathematician Ernest Tilden Parker constructed two orthogonal Latin squares of order 10, and in 1960 Bose, Shrikhande and Parker finally disproved Eulers 177-year old conjecture for all other orders [5,6]. This paper utilizes a set of Mutually Orthogonal Latin Squares (MOLS) of order 7 to construct a series of incomplete block designs, such as; balanced incomplete block designs (BIBD), partially balanced incomplete block designs of  $m$  associate classes (PBIBD ( $m$ )) and Near-Resolvable BIBDs.

## Latin Squares

A latin square of order  $n$  is an  $n$  by  $n$  array in which  $n$  distinct symbols from a symbol set  $n$ , say  $\{1, 2, \dots, n\}$  are arranged, such that each symbol occurs exactly once in each row and in each column.

### Example 1

Latin squares of order 3 ( $L_3$ ), 4 ( $L_4$ ) and 5 ( $L_5$ ) are given below:

Given below as  $L_1, L_2,$  and  $L_3,$  are Latin squares of orders 3, 4 and 5 respectively.

$L_1 =$

1	2	3
2	3	1
3	1	2

$L_2 =$

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

$L_3 =$

5	1	2	3	4
2	3	4	5	1
4	5	1	2	3
1	2	3	4	5
3	4	5	1	2

**Definition 1** (Reduced or Standard Latin Square)

A reduced or standard Latin square of order  $n$  is a Latin square that has its first row and column in the natural order  $(1,2,3,\dots,n)$ .

**Example 2**

Reduced Latin square of order 4. Given here as  $L_2,$  is a reduced Latin square of order 4.

$L_2 =$

1	2	3	4
2	1	4	3
3	4	2	1
4	3	1	2

**Definition 2** (Transpose of a Latin Square)

The transpose of a latin square  $L$  of order  $n$  on the symbol set  $S,$  denoted by  $L^T,$  is latin square defined by  $L^T(i,j)=L(j,i)$  for all  $1 \leq i, j \leq n$

**Definition 3** (Symmetric Latin Square)

A Latin square  $L$  of order  $n$  is symmetric if  $L(i, j) = L(j, i)$  for all  $1 \leq i, j \leq n$

**Example 3**

Symmetric Latin square of order  $n = 5$

Given here as  $L$  and  $L^T$  are symmetric matrix and  $L$  its transpose  $L^T$

$L =$

5	1	2	3	4
1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3

And

$L^T =$

5	1	2	3	4
1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3

**Definition 4** (Idempotent Latin Square)

A Latin square  $L$  order  $n$  is idempotent if  $L(i,j) = i$  for  $1 \leq i, j \leq n$  or equivalently if the main diagonal of the Latin square is a transversal in natural order.

$L =$

1	7	6	5	4	3	2
3	2	1	7	6	5	4
5	4	3	2	1	7	6
7	6	5	4	3	2	1
2	1	7	6	5	4	3
4	3	2	1	7	6	5
6	5	4	3	2	1	7

**Definition 5** (Mutually Orthogonal Latin squares)

Two Latin squares  $L_1$  and  $L_2$  of the same order, say  $n$  are mutually orthogonal if every ordered pair  $(i, j) 1 \leq i, j \leq n$  appears exactly once when  $L_1$  and  $L_2$  are superimposed on each other.

**Example 4**

Mutually Orthogonal Latin Square of Order 4

Given here is a pair of orthogonal Latin square of order 4.

$L_1 =$

4	1	2	3
1	4	3	2
2	3	4	1
3	2	1	4

$L_2 =$

4	1	2	3
3	2	1	4
1	4	3	3
2	3	4	1

And

$S(L_1, L_2) =$

(4,4)	(1,1)	(2,2)	(3,3)
(1,3)	(4,2)	(3,1)	(2,4)
(2,1)	(3,4)	(4,3)	(1,2)
(3,2)	(2,3)	(1,4)	(4,1)

**Mutually Orthogonal Latin Squares of Order 7**

Here a complete set of mutually orthogonal Latin squares of order  $n = 7$  which subsequently serves as a base for the construction of balanced incomplete block designs is presented. Meanwhile, a set of mutually orthogonal Latin squares is constituted by two or

more latin squares of the same order, all of which are orthogonal to one another [7]. Furthermore, the sub (super) diagonal of a square is the set of elements directly below (above) the elements comprising the diagonal.

**Example 5**

A complete set of MOLS of order 7

A complete set of mutually orthogonal Latin squares of order 7 is presented below:

$L_1 =$

7	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	1
3	4	5	6	7	1	2
4	5	6	7	1	2	3
5	6	7	1	2	3	4
6	7	1	2	3	4	5

$L_2 =$

7	1	2	3	4	5	6
2	3	4	5	6	7	1
4	5	6	7	1	2	3
6	7	1	2	3	4	5
1	2	3	4	5	6	7
3	4	5	6	7	1	2
5	6	7	1	2	2	4

$L_3 =$

7	1	2	3	4	5	6
3	4	5	6	7	1	2
6	7	1	2	3	4	5
2	3	4	5	6	7	1
5	6	7	1	2	3	4
1	2	3	4	5	6	7
4	5	6	7	1	2	3

$L_4 =$

7	1	2	3	4	5	6
4	5	6	7	1	2	3
1	2	3	4	5	6	7
5	6	7	1	2	3	4
2	3	4	5	6	7	1
6	7	1	2	3	4	5
3	4	5	6	7	1	2

$L_5 =$

7	1	2	3	4	5	6
5	6	7	1	2	3	4
3	4	5	6	7	1	2
1	2	3	4	5	6	7
6	7	1	2	3	4	5
4	5	6	7	1	2	3
2	3	4	5	6	7	1

$L_6 =$

7	1	2	3	4	5	6
6	7	1	2	3	4	5
5	6	7	1	2	3	4
4	5	6	7	1	2	3
3	4	5	6	7	1	2
2	3	4	5	6	7	1
1	2	3	4	5	6	7

Note that, since Latin squares  $L_1, L_2, \dots, L_6$  are pairwise orthogonal, they are therefore said to constitute a complete set of MOLS of order 7.

**Definition 6** (A Complete Set of MOLS)

A set  $t \geq 2$  MOLS of order  $n$  is called a complete set if  $t = n-1$  or a set of  $n-1$  MOLS of order  $n$  is called a complete set of MOLS.

**Theorem 1** for  $N(n) \leq n-1$  any  $n \geq 2$  or for each  $n \geq 2, N(n) \leq n-1$ , where  $n$  is the order of the Latin square while  $n-1$ , is the complete set of MOLS.

**Theorem 2** Let  $L_i (i=1,2,\dots,6)$  be a set of orthogonal Latin squares of order 7, if an initial block of size 3 is formed with the upper diagonal of length 2 or 3 of  $L_1$  and  $L_2$  and complementary block ( mod 7 ) are generated with this initial block then the design obtained is a BIBD.

**Theorem 3** Let  $L_i (i=1,2,\dots,6)$  be a set of orthogonal Latin squares of order 7 and let the main  $S_{\pm k}, 1 \leq k \leq 5$  denotes the set of elements on the main diagonals with distance  $\pm k$  from the main diagonal of  $L_i$ . Define the sets  $D_{u,i} = (S_{k,i}), D_{l,i} = (S_{-k,i})$  a collection of the sets  $S_{\pm k}$ .

Case 1 (Latin Square on itself) for a fixed  $i$ , define  $(D_{u,i}, D_{l,i}) := D_{u,i} \cup D_{l,i}, 1 \leq i \leq 7$ .

Case 2 (General) for each  $i, (1 \leq i \leq 7)$ , and  $(1 \leq j \leq 7), j \neq i$  defines

$$(D_{u,i}, D_{l,j}) := [D_{u,i} \cup D_{u,j}, D_{u,i} \cup D_{l,j}, D_{l,i} \cup D_{u,j}, D_{l,i} \cup D_{l,j}]$$

the pattern formed by the combinations  $(D_{u,i}, D_{l,j})$  in both cases is of the form BIBD provided the both conditions below are satisfied:

$$rv = kb \tag{1}$$

$$\lambda(v - 1) = r(k - 1) \tag{2}$$

Using Theorem 3 the following initial blocks were obtained:

$$L_{2u} = (5,1) \text{ and } L_{2l} = (3,6) \text{ then } (L_2, L_2) = (5,1,3,6)$$

$$L_{2u} = (4,7,3) \text{ and } L_{2l} = (3,6) \text{ then } (L_2, L_2) = (4,7,3,6)$$

$$L_{1u} = (5,7) \text{ and } L_{2u} = (5,1) \text{ then } (L_1, L_2) = (5,7,1)$$

$$L_{2u} = (4,7,3) \text{ and } L_{3l} = (5,2,6) \text{ then } (L_2, L_3) = (4,7,3,5,2,6)$$

### Construction of Balanced Incomplete Block Designs (BIBD)

In this section a full specification of four of the several incomplete block designs constructed is effected, by the cyclic development of the appropriate initial blocks. A summary of several other designs that were constructed are hereby presented:

#### Designs

**Design 1:** PBIBD<sub>(3)</sub> with the Initial Block (5,1,3,6)

A partially balanced incomplete block design with three associate class, with parameters

$$v = 7, r = 4, b = 7, k = 4, \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3,$$

is given as follows:

(5, 1, 3, 6)  
 (6, 2, 4, 7)  
 (7, 3, 5, 1)  
 (1, 4, 6, 2)  
 (2, 5, 7, 3)  
 (3, 6, 1, 4)  
 (4, 7, 2, 5)

**Design 2:** PBIBD<sub>(2)</sub> with the Initial Block (4,7,3)

A partially balanced incomplete block design with two associate class, with parameters

$$v = 7, r = 3, b = 7, k = 3, \lambda_1 = 1, \lambda_2 = 2,$$

is given as follows:

(4, 7, 3)  
 (5, 1, 4)  
 (6, 2, 5)  
 (7, 3, 6)  
 (1, 4, 7)  
 (2, 5, 1)  
 (3, 6, 2)

**Design 3:** BIBD with the Initial Block (5, 7, 1)

A balanced incomplete block design with parameters

$$v = 7, r = 3, b = 7, k = 3, \lambda = 1$$

is given as follows:

(5, 7, 1)  
 (6, 1, 2)  
 (7, 2, 3)  
 (1, 3, 4)  
 (2, 4, 5)  
 (3, 5, 6)  
 (4, 6, 7)

**Design 4:** Near Resolvable BIBD with the Initial Block (4, 7, 3, 5, 2, 6)

A near resolvable balanced incomplete block design with parameters

$$v = 7, r = 6, b = 7, k = 6, \lambda = 5$$

is given as follows:

(4, 7, 3, 5, 2, 6)  
 (5, 1, 4, 6, 3, 7)  
 (6, 2, 5, 7, 4, 1)  
 (7, 3, 6, 1, 5, 2)  
 (1, 4, 7, 2, 6, 3)  
 (2, 5, 1, 3, 7, 4)  
 (3, 6, 2, 4, 1, 5)

## Design Evaluation

S/N	$v$	$b$	$r$	$k$	$\lambda$	$\lambda_1$	$\lambda_2$	$\lambda_3$	Efficiency Factor	Initial block
1	7	7	3	3	-	1	2	-		(4, 7, 3)
2	7	7	3	3	-	1	2	-		(4, 2, 7)
3	7	7	3	3	-	1	2	-		(2, 7, 5)
4	7	7	4	4	-	1	2	3		(3, 6, 2, 5)
5	7	7	4	4	-	1	2	3		(5, 1, 3, 6)
6	7	7	3	3	1	-	-	-	0.78	(5, 7, 1)
7	7	7	3	3	1	-	-	-	0.78	(5, 7, 4)
8	7	7	3	3	1	-	-	-	0.78	(3, 6, 4)
9	7	7	4	4	2	-	-	-	0.88	(5, 7, 4, 3)
10	7	7	4	4	2	-	-	-	0.88	(4, 6, 1, 7)
11	7	7	6	6	5	-	-	-	0.97	(4, 6, 1, 2, 7, 5)
12	7	7	6	6	5	-	-	-	0.97	(4, 7, 3, 5, 2, 6)

Table 1: Summary of the Designs Constructed on BIBDs,  $v = 7, 3 \leq r \leq 6$

## Conclusion

The set of Mutually Orthogonal Latin Squares (MOLS) of order 7 was utilized to obtain series of initial blocks which were made to undergo the operation of cyclic development, such that the resulting values satisfied the arithmetic operation in modulo 7. This method indeed gave rise to a new method of construction of incomplete block designs, such as balanced incomplete block designs, and partially balanced incomplete block designs, with  $m$  associate classes, for  $m = 1, 2, 3$  (BIBD, PBIBD (2) and PBIBD (3)) with  $k = 3, 4, 5$  and 6. Also, two distinct Near-Resolvable BIBDs that are symmetric were also constructed. All the designs constructed are with fewer numbers of blocks when compared.

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